A short proof of an Erdős-Ko-Rado theorem for compositions

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Abstract

If \(a_1, \ldots, a_k\) and \(n\) are positive integers such that \(n = a_1 + \cdots + a_k\), then the tuple \((a_1, \ldots, a_k)\) is a composition of \(n\) of length \(k\). We say that \((a_1, \ldots, a_k)\) strongly \(t\)-intersects \((b_1, \ldots, b_k)\) if there are at least \(t\) distinct indices \(i\) such that \(a_i = b_i\). A set \(A\) of compositions is strongly \(t\)-intersecting if every two members of \(A\) strongly \(t\)-intersect. Let \(C_{n,k}\) be the set of all compositions of \(n\) of length \(k\). Ku and Wong [7] proved a short proof of this analogue of the Erdős-Ko-Rado Theorem. It yields an improved value of \(n_0(k,t)\).

1 Introduction

Recently, Ku and Wong [18] proved an analogue of the classical Erdős-Ko-Rado Theorem [7] for weak compositions. In this note we provide a short proof of their result. We set up the main definitions and notation before stating the result.

Unless otherwise stated, we will use small letters such as \(x\) to denote non-negative integers or functions or elements of a set, capital letters such as \(X\) to denote sets, and calligraphic letters such as \(F\) to denote families (that is, sets whose members are sets themselves). We call a set \(A\) an \(r\)-element set if its size \(|A|\) is \(r\). The family of all subsets of a set \(X\) is denoted by \(2^X\), and the family of all \(r\)-element subsets of \(X\) is denoted by \(\binom{X}{r}\). For any integer \(n \geq 1\), the set \(\{1, \ldots, n\}\) of the first \(n\) positive integers is denoted by \([n]\).

If \(a_1, \ldots, a_k\) and \(n\) are positive integers such that \(n = \sum_{i=1}^{k} a_i\), then the \(k\)-tuple \((a_1, \ldots, a_k)\) is a composition of \(n\) of length \(k\). If \(a_1, \ldots, a_k\) and \(n\) are non-negative integers such that \(n = \sum_{i=1}^{k} a_i\), then \((a_1, \ldots, a_k)\) is a weak composition of \(n\) of length \(k\). Let \(C_{n,k}\) be the set of all compositions of \(n\) of length \(k\), and let \(W_{n,k}\) be the set of all weak compositions of \(n\) of length \(k\). An elementary counting result is that \(|W_{n,k}| = \binom{n+k-1}{n}\).

Since \(W_{n-k,k} = \{(a_1 - 1, \ldots, a_k - 1) : (a_1, \ldots, a_k) \in C_{n,k}\}\), \(|C_{n,k}| = \binom{n-1}{n-k}\).
We say that \((a_1, \ldots, a_k)\) strongly \(t\)-intersects \((b_1, \ldots, b_k)\) if there exists \(T \in \binom{[k]}{t}\) such that \(a_i = b_i\) for each \(i \in T\). We call a set \(A\) of \(k\)-tuples strongly \(t\)-intersecting if every two members of \(A\) strongly \(t\)-intersect.

Recently, Ku and Wong [18] proved the following result.

**Theorem 1.1 ([18])** For every two positive integers \(k\) and \(t\) with \(k \geq t + 2\), there exists an integer \(n_0(k, t)\) such that for \(n \geq n_0(k, t)\), the size of every strongly \(t\)-intersecting subset \(A\) of \(W_{n,k}\) is at most \(\left(\frac{n+k-t-1}{n}\right)\), and the bound is attained if and only if for some \(T \in \binom{[n]}{t}\), \(A = \{(a_1, \ldots, a_k) \in W_{n,k} : a_i = 0\text{ for each } i \in T\}\).

We provide a short proof of this result. It yields an improved value of \(n_0(k, t)\). Let \(c(k, t) = (k - t - 1)\binom{3k - 2t - 1}{t+1} + t + 2\).

**Theorem 1.2** If \(t \geq 1, k \geq t + 2, n \geq c(k, t)\), and \(A\) is a strongly \(t\)-intersecting subset of \(C_{n,k}\), then
\[|A| \leq \binom{n-t-1}{n-k},\]
and equality holds if and only if for some \(T \in \binom{[n]}{t}\), \(A = \{(a_1, \ldots, a_k) \in C_{n,k} : a_i = 0\text{ for each } i \in T\}\).

This gives Theorem 1.1 as follows. Let \(t \geq 1, k \geq t + 2\), and \(n \geq c(k, t) - k\). Let \(A\) be a strongly \(t\)-intersecting subset of \(W_{n,k}\), and let \(A' = \{(a_1+1, \ldots, a_k+1) : (a_1, \ldots, a_k) \in A\}\). So \(A'\) is a strongly \(t\)-intersecting subset of \(C_{n',k}\), where \(n' = n + k \geq c(k, t)\). By Theorem 1.2, \(|A'| \leq \binom{n'-t-1}{n'-k} = \binom{n+k-t-1}{n}\), and equality holds if and only if for some \(T \in \binom{[n]}{t}\), \(A' = \{(a_1', \ldots, a_k') \in C_{n',k} : a_i' = 1\text{ for each } i \in T\}\). So \(|A| \leq \binom{n+k-t-1}{n}\), and equality holds if and only if for some \(T \in \binom{[n]}{t}\), \(A = \{(a_1, \ldots, a_k) \in W_{n,k} : a_i = 0\text{ for each } i \in T\}\).

By a similar argument, Theorem 1.1 implies Theorem 1.2 for \(n \geq n_0(k, t) + k\).

The problem is trivial for \(t \leq k \leq t+1\). Let \(A\) be a strongly \(t\)-intersecting subset of \(C_{n,k}\). If \(k = t\), then \(A\) can only have one element. If \(k = t+1\) and \((a_1, \ldots, a_k, b_1, \ldots, b_k) \in A\), then for some \(T \in \binom{[n]}{t}\), we have \(a_i = b_i\) for \(i \in T\). Only one index is outside \(T\). Since \((a_1, \ldots, a_k)\) and \((b_1, \ldots, b_k)\) have the same sum \(n\), they must therefore also agree in the remaining position, so \(|A| = 1\).

The value of \(n_0(k, t)\) obtained in [18] for Theorem 1.1 is \(\max\{(k - t - 1)\binom{k}{t}^2, (2k - 2t)k^{2k-t+1} + 1\}\). As we pointed out above, Theorem 1.1 holds with \(n \geq c(k, t) - k\). It follows that Theorem 1.1 holds with \(n \geq (k - t - 1)(3k - 2t - 1)^{t+1}\).

The dependence of \(n\) on \(t\) in Theorem 1.2 can be avoided by taking \(n\) to be sufficiently large. A crude way of showing this is that \(c(k, t) \leq k^{(3k-2t-1)/(t+1)} < k^{(3k)/(3k/2)}\); so the result is true for \(n \geq k(\frac{3k}{3k/2})\). In Section 3 we show that the dependence of \(n\) on \(k\) is inevitable and that we cannot even replace \(n \geq c(k, t)\) by \(k \geq k_0(t)\).

Theorems 1.1 and 1.2 are analogues of the classical Erdős-Ko-Rado (EKR) Theorem [7], which inspired many results in extremal set theory (see [6, 10, 8, 3]). A family \(\mathcal{A}\) of sets is \(t\)-intersecting if every two sets in \(\mathcal{A}\) have at least \(t\) common elements. The EKR Theorem says that for \(1 \leq t \leq k\), there exists an integer \(n_0(k, t)\) such that for \(n \geq n_0(k, t)\), the size of any \(t\)-intersecting subfamily of \(\binom{[n]}{k}\) is at most \(\binom{n-t}{k-t}\), which is the size of the simplest \(t\)-intersecting subfamily \(\{A \in \binom{[n]}{k} : |t| \leq A\}\). It was also shown in [7] that the smallest possible value of \(n_0(k, 1)\) is \(2k\). There are various proofs of this (see [16, 11, 14, 5]), two of which are particularly short and beautiful: Katona’s [14], introducing the elegant cycle method, and Daykin’s [5], using the powerful Kruskal-Katona Theorem [15, 17]. Frankl [9] showed that for \(t \geq 15\), the smallest possible value
of \( n_0(k, t) \) is \((k - t + 1)(t + 1)\). Subsequently, Wilson [20] proved this for all \( t \geq 1 \). Frankl [9] conjectured that the size of a largest \( t \)-intersecting subfamily of \( \binom{n}{k} \) is \( \max \{|A|: |A \cap [t + 2i]| \geq t + i, i \in \{0\} \cup [k - t]\} \). A remarkable proof of this conjecture together with the complete characterisation of the extremal structures was obtained by Ahlswede and Khachatrian [1]. The \( t \)-intersection problem for \( 2^n \) was completely solved by Katona [16]. These are prominent results in extremal set theory.

As will become clearer in the proof, Theorem 1.2 can also be phrased in terms of \( t \)-intersecting subfamilies of a family. Indeed, it is equivalent to the following: if \( n \geq c(k, t) \) and \( A \) is a \( t \)-intersecting subfamily of the family \( \mathcal{C}_{n,k} = \{\{(1, a_1), \ldots, (k, a_k)\} : (a_1, \ldots, a_k) \in C_{n,k}\} \), then \(|A| \leq \binom{n-1}{n-k} \), and equality holds if and only if for some \( T \in \binom{k}{t} \), \( A = \{\{(1, a_1), \ldots, (k, a_k)\} \in C_{n,k} : a_i = 1 \text{ for each } i \in T\} \).

EKR-type results have been obtained in a wide variety of contexts, many of which are surveyed in [6, 10, 8, 3]. Usually the objects have symmetry properties (see [4, Section 3.2] and [19]) or enable use of compression operators (also called shift operators) to push \( t \)-intersecting families towards a desired form (see [10, 13, 12]). One of the main motivating factors behind this note is that although the family \( C_{n,k} \) does not have any of these properties, we can still determine its largest \( t \)-intersecting subfamilies for \( n \) sufficiently large, using more than one method. It is interesting that Ku and Wong [18] managed to take an inductive approach. We will show that the method in [7] can be adapted to this framework. However, since \( C_{n,k} \) does not have any of the above properties, the problem of determining the maximum size of a \( t \)-intersecting subfamily of \( C_{n,k} \) for any \( n, k \) and \( t \) must be very hard. We conjecture that the extremal structures are similar to those in the above-mentioned conjecture of Frankl (proved in [1]). We state the conjecture using the original formulation.

**Conjecture 1.3** Let \( 1 \leq t \leq k \leq n \). For \( i = 0, \ldots, \lfloor \frac{k-t}{2} \rfloor \), let \( A_i = \{(a_1, \ldots, a_k) \in C_{n,k} : |\{j \in [t+2i] : a_j = 1\}| \geq t+i\} \). The size of a largest strongly \( t \)-intersecting subset of \( C_{n,k} \) is \( \max \{|A_i| : 0 \leq i \leq \lfloor \frac{k-t}{2} \rfloor\} \).

# 2 Proof of Theorem 1.2

A \( t \)-intersecting family is non-trivial if its members have fewer than \( t \) common elements.

The following lemma emerges from [7] (see also [2, Proof of Theorem 2.1]).

**Lemma 2.1** If \( A \) is a non-trivial \( t \)-intersecting family whose members are of size at most \( k \), then there exists a set \( J \) of size at most \( 3k - 2t - 1 \) such that \(|A \cap J| \geq t + 1 \) for each \( A \in \mathcal{A} \).

This lemma is the key ingredient of the proof of Theorem 1.2, which we can now provide. As indicated in Section 1, we transform the setting of compositions to a setting of sets of pairs.

**Proof of Theorem 1.2.** Let \( k \geq t + 2 \) and \( n \geq c(k, t) \). Let \( A \) be a non-empty strongly \( t \)-intersecting subset of \( C_{n,k} \). Write \( a \) for a composition \((a_1, \ldots, a_k)\). Let \( S_a = \{(i, a_i) : i \in [k]\} \). Let \( C_{n,k} = \{S_a : a \in C_{n,k}\} \). Let \( f : C_{n,k} \to C_{n,k} \) such that \( f(a) = S_a \) for each \( a \in C_{n,k} \). Clearly, \( f \) is a bijection. Note that two compositions \( a \) and \( b \) strongly \( t \)-intersect if and only if \(|S_a \cap S_b| \geq t\). Thus, a subset \( I \) of \( C_{n,k} \) is strongly \( t \)-intersecting if and only if \( \{S_a : a \in I\} \) is a strongly \( t \)-intersecting subfamily of \( C_{n,k} \).

Letting \( \mathcal{A} = \{f(a) : a \in A\} \), we have that \(|\mathcal{A}| = |A|\), \( \mathcal{A} \) is a \( t \)-intersecting subfamily of \( C_{n,k} \), and \(|X| = k\) for each \( X \in \mathcal{A} \).
Suppose that the sets in $\mathcal{A}$ have $t$ common elements $(h_1, d_{h_1}), \ldots, (h_t, d_{h_t})$. Let $D = \{(a_1, \ldots, a_k) \in C_{n,k}: a_i = d_{h_i} \text{ for each } i \in [t]\}$. Thus $A \subseteq D$. Let $p = \sum_{i=1}^{t} d_{h_i}$. Note that $|D| = |C_{n-p,k-t}| = \binom{n-p-1}{n-p-k+t}$. Since $d_{h_i} \geq 1$ for each $i \in [t]$, we have $p \geq t$. Hence $|D| \leq \binom{n-t-1}{n-t-k}$, and equality holds if and only if $p = t$. Now $p = t$ if and only if $d_{h_i} = 1$ for each $i \in [t]$. Thus $|A| \leq \binom{n-t-1}{n-t-k}$, and equality holds if and only if $A = \{(a_1, \ldots, a_k) \in C_{n,k}: a_i = 1 \text{ for each } i \in [t]\}$.

Now suppose that the sets in $\mathcal{A}$ do not have $t$ common elements, so $\mathcal{A}$ is a non-trivial $t$-intersecting family. By Lemma 2.1, there exists a set $J$ such that $|J| \leq 3k - 2t - 1$ and $|X \cap J| \geq t + 1$ for each $X \in \mathcal{A}$. Thus $\mathcal{A} \subseteq \bigcup_{T \in \binom{J}{t+1}} \{X \in C_{n,k}: T \subset X\}$. Let $T^* \in \binom{J}{t+1}$ such that $|\{X \in C_{n,k}: T^* \subset X\}| \leq |\{X \in C_{n,k}: T \subset X\}|$ for all $T \in \binom{J}{t+1}$. Let $\mathcal{B} = \{X \in C_{n,k}: T^* \subset X\}$. We have

$$|A| \leq \left| \bigcup_{T \in \binom{J}{t+1}} \{X \in C_{n,k}: T \subset X\} \right| \leq \sum_{T \in \binom{J}{t+1}} |\{X \in C_{n,k}: T \subset X\}| \leq \sum_{T \in \binom{J}{t+1}} |\mathcal{B}| = \left| C_{n,q-k-(t+1)} \right| = \binom{n-q-1}{n-q+k+(t+1)} \leq \binom{n-(t+1)-1}{n-(t+1)-k+(t+1)} = \binom{n-t-2}{n-k}.$$ 

Hence $|A| \leq |A| \leq \binom{3k-2t-1}{t+1}|\mathcal{B}| \leq \binom{3k-2t-1}{t+1} \binom{n-t-2}{n-k}$. Now $\binom{n-t-1}{n-k} = \frac{n-t-1}{n-k} \binom{n-t-2}{n-k}$. Thus, since $n \geq c(k,t)$, we have $\binom{n-t-1}{n-k} > \binom{3k-2t-1}{t+1} \binom{n-t-2}{n-k}$, and $|A| < \binom{n-t-1}{n-k}$.

\section{Dependence on $k$}

We now show that the dependence of $n$ on $k$ is inevitable. Let $T_{n,k} = \{(a_1, \ldots, a_k) \in C_{n,k}: a_i = 1 \text{ for each } i \in [t]\}$ and $N_{n,k} = \{(a_1, \ldots, a_k) \in C_{n,k}: \{|i \in [t+2]: a_i = 1\} \geq t+1\}$. Note that $T_{n,k}$ and $N_{n,k}$ are strongly $t$-intersecting subsets of $C_{n,k}$, and $T_{n,k}$ is one of the optimal families given by Theorem 1.2. We have

$$|N_{n,k}| - |T_{n,k}| = ((t+2)C_{n-t-1,k-t-1}) - (t+1)C_{n-t-2,k-t-2}) - |C_{n-t,k-1}|$$

$$= (t+2)\binom{n-t-2}{n-k} - (t+1)\binom{n-t-3}{n-k} - \binom{n-t-1}{n-k}$$

$$= (\binom{n-t-2}{n-k} (t+2) - (t+1)\binom{k-t-2}{k-t-1} - \binom{n-t-1}{n-k})$$

$$= (\binom{n-t-2}{n-k} \frac{(t+1)(n-k)}{n-t-2} - \binom{n-t-1}{n-k})$$

$$= (n-k) \binom{n-t-2}{n-k} \frac{(t+1)(k-t) + 1 - n}{(n-t-2)(k-t-1)}.$$ 

Thus, $|N_{n,k}| > |T_{n,k}|$ if $k + 1 \leq n \leq (t+1)(k-t)$. No matter how large $k$ or $n$ is, Theorem 1.2 does not hold if $k+1 \leq n \leq (t+1)(k-t)$. In other words, we cannot replace $n \geq c(k,t)$ by $n \geq n_0(t)$ or $k \geq k_0(t)$.

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References


