

Isolation of non-triangle cycles in graphs

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Abstract

Given a set $\mathcal{F}$ of graphs, we call a copy of a graph in $\mathcal{F}$ an $\mathcal{F}$ -graph. The $\mathcal{F}$ -isolation number of a graph G , denoted by $\iota(G, \mathcal{F})$, is the size of a smallest set D of vertices of G such that the closed neighbourhood of D intersects the vertex sets of the $\mathcal{F}$ -graphs contained by G (equivalently, $G - N[D]$ contains no $\mathcal{F}$ -graph). Let $\mathcal{C}$ be the set of cycles, and let $\mathcal{C}'$ be the set of non-triangle cycles (that is, cycles of length at least 4). Let G be a connected graph having exactly n vertices and m edges. The first author proved that $\iota(G, \mathcal{C}) \leq n/4$ if G is not a triangle. Bartolo and the authors proved that $\iota(G, \{C_4\}) \leq n/5$ if G is not a copy of one of nine graphs. Various authors proved that $\iota(G, \mathcal{C}) \leq (m+1)/5$ if G is not a triangle. We prove that $\iota(G, \mathcal{C}') \leq (m+1)/6$ if G is not a 4-cycle. Zhang and Wu established this for the case where G is triangle-free. Our result yields the inequality $\iota(G, \{C_4\}) \leq (m+1)/6$ of Wei, Zhang and Zhao. These bounds are attained by infinitely many (non-isomorphic) graphs. The proof of our inequality hinges on also determining the graphs attaining the bound.

1 Introduction

Unless stated otherwise, we use small letters such as x to denote non-negative integers or elements of sets, and capital letters such as X to denote sets or graphs. For $n \geq 0$, $[n]$ denotes the set $\{i \in \mathbb{N} : i \leq n\}$, where $\mathbb{N}$ is the set of positive integers. Note that $[0]$ is the empty set $\emptyset$. Arbitrary sets are taken to be finite. For a set X , $\binom{X}{2}$ denotes the set of 2-element subsets of X . We may represent a 2-element set $\{x, y\}$ by xy .

For standard terminology in graph theory, we refer the reader to [23]. Most of the notation and terminology used here is defined in [2], which motivates the work in this paper.

Every graph G is taken to be *simple*, that is, G is a pair $(V(G), E(G))$ such that $V(G)$ and $E(G)$ (the vertex set and the edge set of G) are sets that satisfy $E(G) \subseteq \binom{V(G)}{2}$. We call G an *n -vertex graph* if $|V(G)| = n$. We call G an *m -edge graph* if $|E(G)| = m$. For a vertex v of G , $N_G(v)$ denotes the set of neighbours of v in G , $N_G[v]$ denotes the closed neighbourhood $N_G(v) \cup \{v\}$ of v , and $d_G(v)$ denotes the

degree $|N_G(v)|$ of v . For a subset X of $V(G)$, $N_G[X]$ denotes the closed neighbourhood $\bigcup_{v \in X} N_G[v]$ of X , $G[X]$ denotes $(X, E(G) \cap \binom{X}{2})$ (the subgraph of G induced by X), and $G - X$ denotes the graph $G[V(G) \setminus X]$ (obtained by deleting the vertices in X from G). Where no confusion arises, the subscript G may be omitted from any notation that uses it; for example, $N_G(v)$ may be abbreviated to $N(v)$. If H is a subgraph of G , then we say that G *contains* H . If F is a copy of G , then we write $F \simeq G$.

For $n \geq 1$, the graphs $([n], \binom{[n]}{2})$ and $([n], \{\{i, i+1\} : i \in [n-1]\})$ are denoted by K_n and P_n , respectively. For $n \geq 3$, C_n denotes the graph $([n], \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\})$. A copy of K_n is called an n -*clique* or a *complete graph*. A copy of P_n is called an n -*path* or simply a *path*. A copy of C_n is called an n -*cycle* or simply a *cycle*. A 3-clique is a 3-cycle and is also called a *triangle*.

If $\mathcal{F}$ is a set of graphs and F is a copy of a graph in $\mathcal{F}$, then we call F an $\mathcal{F}$ -*graph*. A subset D of $V(G)$ is called an $\mathcal{F}$ -*isolating set* of G if D intersects the vertex sets of the $\mathcal{F}$ -graphs contained by G . Thus, D is an $\mathcal{F}$ -isolating set of G if and only if $G - N[D]$ contains no $\mathcal{F}$ -graph. It is to be assumed that $(\emptyset, \emptyset) \notin \mathcal{F}$. The size of a smallest $\mathcal{F}$ -isolating set of G is denoted by $\iota(G, \mathcal{F})$ and is called the $\mathcal{F}$ -*isolation number* of G . If $\mathcal{F} = \{F\}$, then we may replace $\mathcal{F}$ in these defined terms and notation by F .

The study of isolating sets was initiated by Caro and Hansberg [10]. It generalizes the study of the classical domination problem [13, 14, 16, 17, 18, 19] naturally. Indeed, D is a *dominating set* of G (that is, $N[D] = V(G)$) if and only if D is a K_1 -isolating set of G , so the *domination number* is the K_1 -isolation number. One of the earliest domination results is the upper bound $n/2$ of Ore [21] on the domination number of any connected n -vertex graph $G \not\simeq K_1$ (see [16]). While deleting the closed neighbourhood of a dominating set produces the graph with no vertices, deleting the closed neighbourhood of a K_2 -isolating set produces a graph with no edges. In the literature, a K_2 -isolating set is also called a *vertex-edge dominating set*. Caro and Hansberg [10] proved that if G is a connected n -vertex graph with $n \geq 3$, then $\iota(G, K_2) \leq n/3$ unless G is a 5-cycle. This was independently proved by Żyliński [27] and solved a problem in [8]. Fenech, Kaemawichanurat and the first author of this paper [6] generalized these bounds by showing that for any $k \geq 1$, $\iota(G, K_k) \leq n/(k+1)$ unless $G \simeq K_k$ or $k = 2$ and G is a 5-cycle. This sharp bound settled a problem of Caro and Hansberg [10]. The graphs attaining the bound are determined in [9, 11, 12, 20]. Fenech, Kaemawichanurat and the first author [7] also showed that $\iota(G, K_k) \leq (m+1)/(\binom{k}{2} + 2)$ unless $G \simeq K_k$, and they determined the graphs attaining the bound. Generalizations of these bounds are given in [3, 5].

Let $\mathcal{C}$ be the set of cycles. The first author [2] obtained the following bound on $\iota(G, \mathcal{C})$, and consequently settled another problem of Caro and Hansberg [10].

Theorem 1 ([2]) *If G is a connected n -vertex graph that is not a triangle, then*

$$\iota(G, \mathcal{C}) \leq \frac{n}{4}.$$

Moreover, the bound is sharp.

He also gave an explicit construction of a connected n -vertex graph that attains the bound $\lfloor n/4 \rfloor$ resulting from Theorem 1. The graphs that attain the bound $n/4$ in the

theorem are determined in [11]. Various authors have obtained an analogue of Theorem 1 that provides a sharp bound on $\iota(G, \mathcal{C})$ in terms of the number of edges. In order to state the full result, we need the following construction [4, Construction 1], which is a generalization of [7, Construction 1.2] and a slight variation of the construction of $B_{n,F}$ in [2].

Construction 1 ([4]) Consider any $m, k \in \{0\} \cup \mathbb{N}$ and any connected k -edge graph F , where $F \simeq K_1$ if $k = 0$ (that is, $V(F) \neq \emptyset$). By the division algorithm, there exist $q, r \in \{0\} \cup \mathbb{N}$ such that $m + 1 = q(k + 2) + r$ and $0 \leq r \leq k + 1$. Let $Q_{m,k}$ be a set of size q . If $q \geq 1$, then let $v_1, \dots, v_q$ be the elements of $Q_{m,k}$, let $F_1, \dots, F_q$ be copies of F such that the $q + 1$ sets $V(F_1), \dots, V(F_q)$ and $Q_{m,k}$ are pairwise disjoint, and for each $i \in [q]$, let $w_i \in V(F_i)$, and let G_i be the graph with $V(G_i) = \{v_i\} \cup V(F_i)$ and $E(G_i) = \{v_i w_i\} \cup E(F_i)$. If either $q = 0$, T is the null graph $(\emptyset, \emptyset)$, and G is a connected m -edge graph T' , or $q \geq 1$, T is a tree with vertex set $Q_{m,k}$ (so $|E(T)| = q - 1$), T' is a connected r -edge graph with $V(T') \cap \bigcup_{i=1}^q V(G_i) = \{v_q\}$, and G is a graph with $V(G) = V(T') \cup \bigcup_{i=1}^q V(G_i)$ and $E(G) = E(T) \cup E(T') \cup \bigcup_{i=1}^q E(G_i)$, then we say that G is an (m, F) -special graph with quotient graph T and remainder graph T' , and for each $i \in [q]$, we call G_i an F -constituent of G , and we call v_i the F -connection of G_i in G . We say that an (m, F) -special graph is *pure* if its remainder graph has no edges ([7, Figure 1] is an illustration of a pure $(71, K_5)$ -special graph). Clearly, an (m, F) -special graph is a connected m -edge graph.

Theorem 2 ([5, 15, 25]) *If G is a connected m -edge graph that is not a triangle, then*

$$\iota(G, \mathcal{C}) \leq \frac{m + 1}{5}.$$

Moreover, equality holds if and only if G is a pure (m, C_3) -special graph or a 4-cycle.

Theorem 1 has inspired many other results. Consider a connected graph G , and let $n = |V(G)|$ and $m = |E(G)|$. Bartolo and the present authors [1] proved that $\iota(G, C_4) \leq n/5$ if G is not a copy of one of nine particular graphs. This implies the result in [24]. Suppose that G is not a 4-cycle. Wei, Zhang and Zhao [22] showed that

$$\iota(G, \mathcal{F}) \leq \frac{m + 1}{6} \tag{1}$$

if $\mathcal{F} = \{C_4\}$. Zhang and Wu [26] showed that (1) holds if $\mathcal{F} = \mathcal{C}$ and G contains no triangle. Let $\mathcal{C}'$ be the set of cycles that are not triangles. Thus, $\mathcal{C}' = \{H \in \mathcal{C} : |V(H)| \geq 4\}$ and $C_4 \in \mathcal{C}'$. For the result of Zhang and Wu, we have $\iota(G, \mathcal{F}) = \iota(G, \mathcal{C}')$ due to the condition that G contains no triangle. Generalizing both the Wei–Zhang–Zhao result and the Zhang–Wu result, we show that (1) holds also if this condition is dropped and $\mathcal{F} = \mathcal{C}'$. We also determine the extremal graphs. Let C'_4 be the *diamond graph* ([4], $E(C_4) \cup \{\{1, 3\}\}$). We can now state our result, which is proved in the next section.

Theorem 3 *If G is a connected m -edge graph that is not a 4-cycle, then*

$$\iota(G, \mathcal{C}') \leq \frac{m + 1}{6}. \tag{2}$$

Moreover, the following statements hold:

(i) Equality in (2) holds if and only if G is a pure (m, C_4) -special graph or a $\{C'_4, C_5\}$ -graph.

(ii) If G is an (m, C_4) -special graph, then $\iota(G, \mathcal{C}') = \lfloor (m+1)/6 \rfloor$.

It is worth pointing out that the proof of (2) makes use of (i) in an inductive argument.

2 Proof of Theorem 3

We start the proof of Theorem 3 with two basic lemmas.

Lemma 1 ([2]) *If G is a graph, $\mathcal{F}$ is a set of graphs, $X \subseteq V(G)$ and $Y \subseteq N[X]$, then*

$$\iota(G, \mathcal{F}) \leq |X| + \iota(G - Y, \mathcal{F}).$$

Lemma 2 ([2, 5]) *If $G_1, \dots, G_r$ are the distinct components of a graph G , and $\mathcal{F}$ is a set of connected graphs, then $\iota(G, \mathcal{F}) = \sum_{i=1}^r \iota(G_i, \mathcal{F})$.*

The next lemma concerns a case where no member of a subset Y of $V(G)$ is a vertex of an $\mathcal{F}$ -graph contained by G , where $\mathcal{F}$ is a set of cycles.

Lemma 3 ([1]) *If G is a graph, $\mathcal{F}$ is a set of cycles, $x \in V(G)$, $Y \subseteq V(G) \setminus \{x\}$, $N[Y] \cap V(G - Y) \subseteq \{x\}$, and $G[\{x\} \cup Y]$ contains no $\mathcal{F}$ -graph, then $\iota(G, \mathcal{F}) = \iota(G - Y, \mathcal{F})$ and every $\mathcal{F}$ -isolating set of $G - Y$ is an $\mathcal{F}$ -isolating set of G .*

An *isolated vertex* of G is a vertex of G of degree 0. A *leaf* of G is a vertex of G of degree 1.

Corollary 1 ([1]) *If G is a graph, $\mathcal{F}$ is a set of cycles, and y is an isolated vertex of G or a leaf of G , then $\iota(G, \mathcal{F}) = \iota(G - y, \mathcal{F})$.*

Corollary 1 generalizes as follows.

Corollary 2 *If G is a graph, $\mathcal{F}$ is a set of cycles, and $\emptyset \neq Y \subseteq V(G)$ such that each member of Y is an isolated vertex of G or a leaf of G , then $\iota(G, \mathcal{F}) = \iota(G - Y, \mathcal{F})$.*

Proof. We use induction on $|Y|$. If $|Y| = 1$, then the result is Corollary 1. Suppose $|Y| \geq 2$. Let $y \in Y$. By Corollary 1, $\iota(G - y, \mathcal{F}) = \iota(G, \mathcal{F})$. Let $Y' = Y \setminus \{y\}$. Then, each member of Y' is an isolated vertex of $G - y$ or a leaf of $G - y$. By the induction hypothesis, $\iota(G - y, \mathcal{F}) = \iota((G - y) - Y', \mathcal{F})$. Since $\iota(G - y, \mathcal{F}) = \iota(G, \mathcal{F})$ and $\iota((G - y) - Y', \mathcal{F}) = \iota(G - Y, \mathcal{F})$, the result follows. $\square$

For a vertex v of a graph G , let $E_G(v)$ denote the set $\{vw : w \in N_G(v)\}$. For $X, Y \subseteq V(G)$, let $E_G(X, Y)$ denote the set $\{xy \in E(G) : x \in X, y \in Y\}$. Let $C(G)$ denote the set of components of G .

Lemma 4 *If G is a graph, $\mathcal{F}$ is a set of cycles, $Y \subseteq V(G)$, $G[Y]$ contains no $\mathcal{F}$ -graph, and $|E_G(V(H), Y)| \leq 1$ for each $H \in \mathcal{C}(G - Y)$, then $\iota(G, \mathcal{F}) \leq \iota(G - Y, \mathcal{F})$ and every $\mathcal{F}$ -isolating set of $G - Y$ is an $\mathcal{F}$ -isolating set of G .*

Proof. Suppose that G contains an $\mathcal{F}$ -graph F . Since $V(F) \not\subseteq V(G[Y])$, $V(F) \cap V(H) \neq \emptyset$ for some $H \in \mathcal{C}(G - Y)$. Let $X = V(H)$ and $Z = V(G) \setminus X$. Suppose $V(F) \not\subseteq X$. Then, $z_1 z_2 \in E(F)$ for some $z_1 \in X$ and $z_2 \in Z$. Since $H \in \mathcal{C}(G - Y)$, $E_G(X, Z) = E_G(X, Y)$. Since $|E_G(X, Y)| \leq 1$, $z_2 = y$ for some $y \in Y$, and $E_G(X, Z) = \{z_1 z_2\}$. We have $E(F) = \{z_1 z_2, z_2 z_3, \dots, z_{r-1} z_r, z_r z_1\}$ for some $r \geq 3$ and some distinct members $z_3, \dots, z_r$ of $V(G) \setminus \{z_1, z_2\}$. We have $N_F(y) = N_F(z_2) = \{z_1, z_3\}$. Since $E_G(X, Z) = \{z_1 z_2\}$, $z_3 \notin X \cup \{y\}$. Since $y \notin \{z_3, \dots, z_r\}$ and $E_G(X, Z) = \{z_1 z_2\}$, we obtain $z_3, \dots, z_r, z_1 \notin X \cup \{y\}$, which contradicts $z_1 \in X$. Therefore, $V(F) \subseteq X$. Consequently, if D is an $\mathcal{F}$ -isolating set of $G - Y$, then $\emptyset \neq N_{G-Y}[D] \cap V(F) \subseteq N_G[D] \cap V(F)$, meaning that D is an $\mathcal{F}$ -isolating set of G . $\square$

Proposition 1 (a) *If G is a pure (m, C_4) -special graph with exactly q C_4 -constituents, then $m = 6q - 1$, $\iota(G, \mathcal{C}') = q$, and for any $v \in V(G)$, G has a $\mathcal{C}'$ -isolating set D with $v \in D$ and $|D| = q$.*
(b) *If G is a $\{C'_4, C_5\}$ -graph, then $\iota(G, \mathcal{C}') = 1 = (|E(G)| + 1)/6$, and for any $v \in V(G)$, $\{v\}$ is a $\mathcal{C}'$ -isolating set of G .*

Proof. Suppose that G is a pure (m, F) -special graph with exactly q F -constituents as in Construction 1, where $F = C_4$. For some $j \in [q]$, $v \in V(G_j)$. Let $D = \{v\} \cup \{v_i : i \in [q] \setminus \{j\}\}$. Then, D is a $\mathcal{C}'$ -isolating set of G , so $\iota(G, \mathcal{C}') \leq q$. If S is a $\mathcal{C}'$ -isolating set of G , then, since $G_1 - v_1, \dots, G_q - v_q$ are 4-cycles, $S \cap V(G_i) \neq \emptyset$ for each $i \in [q]$. Therefore, $\iota(G, \mathcal{C}') = q$. Now $m = 5q + |E(T)|$. Since T is a q -vertex tree, $|E(T)| = q - 1$. Thus, $m = 6q - 1$. This settles (a). (b) is trivial. $\square$

Proof of Theorem 3. The argument in the proof of Proposition 1 yields (ii). Proposition 1 settles the sufficiency condition in (i). We now prove (2) and (i), using induction on m .

The result is trivial if $m \leq 4$ as $G \not\cong C_4$. Suppose $m \geq 5$. Let k be the maximum degree $\max\{d(v) : v \in V(G)\}$ of G . Since G is connected, $k \geq 2$. If $k = 2$, then G is a path or a cycle. If G is a path, then $\iota(G, \mathcal{C}') = 0 < (m + 1)/6$. If G is a cycle, then $\iota(G, \mathcal{C}') = 1 \leq (m + 1)/6$, and equality holds only if $m = 5$. Suppose $k \geq 3$. Let $v \in V(G)$ with $d(v) = k$. Suppose $N[v] = V(G)$. Then, $\{v\}$ is a $\mathcal{C}'$ -isolating set of G , so $\iota(G, \mathcal{C}') \leq 1 \leq (m + 1)/6$. If $\iota(G, \mathcal{C}') = (m + 1)/6$, then $G \simeq C'_4$. Now suppose $V(G) \neq N[v]$. Let $G' = G - N[v]$. Then, $V(G') \neq \emptyset$. Let $\mathcal{H} = \mathcal{C}(G')$. Let $\mathcal{H}' = \{H \in \mathcal{H} : H \simeq C_4\}$. For each $H \in \mathcal{H} \setminus \mathcal{H}'$, let D_H be a $\mathcal{C}'$ -isolating set of H of size $\iota(H, \mathcal{C}')$. By the induction hypothesis, for each $H \in \mathcal{H} \setminus \mathcal{H}'$,

$$|D_H| \leq \frac{|E(H)| + 1}{6},$$

and equality holds only if H is a pure $(|E(H)|, C_4)$ -special graph or a $\{C'_4, C_5\}$ -graph.

For any $H \in \mathcal{H}$ and any $x \in N(v)$ such that $xy_{x,H} \in E(G)$ for some $y_{x,H} \in V(H)$, we say that H is *linked to* x and that x is *linked to* H . Since G is connected, each member

H of $\mathcal{H}$ is linked to at least one member of $N(v)$, so $x_H y_H \in E(G)$ for some $x_H \in N(v)$ and some $y_H \in V(H)$. For each $x \in N(v)$, let $\mathcal{H}'_x = \{H \in \mathcal{H}' : H \text{ is linked to } x\}$ and $\mathcal{H}_x^* = \{H \in \mathcal{H} \setminus \mathcal{H}' : H \text{ is linked to } x \text{ only}\}$.

Let $H_1, \dots, H_p$ be the distinct members of $\mathcal{H}$. For each $i \in [p]$, let $m_i = |E(H_i)|$. Let $I = \{i \in [p] : H_i \text{ is a pure } (m_i, C_4)\text{-special graph}\}$. For each $i \in I$, let $G_{i,1}, \dots, G_{i,q_i}$ be the C_4 -constituents of H_i , and for each $j \in [q_i]$, let $v_{i,j}$ be the C_4 -connection of $G_{i,j}$ in H_i , and let $w_{i,j}^1, \dots, w_{i,j}^4$ be the members of $V(G_{i,j}) \setminus \{v_{i,j}\}$ such that $E(G_{i,j}) = \{v_{i,j}w_{i,j}^1, w_{i,j}^1w_{i,j}^2, w_{i,j}^2w_{i,j}^3, w_{i,j}^3w_{i,j}^4, w_{i,j}^4w_{i,j}^1\}$. By Proposition 1, $m_i + 1 = 6q_i$ for each $i \in I$. Let $I' = \{i \in [p] : H_i \text{ is a } \{C'_4, C_5\}\text{-graph}\}$. Then, $m_i + 1 = 6$ for each $i \in I'$. Let $J = \{i \in [p] : H_i \simeq C_4\}$. Let $J' = [p] \setminus (I \cup I' \cup J)$. Then, I, I', J and J' are pairwise disjoint, $\mathcal{H}' = \{H_i : i \in J\}$ and $\mathcal{H} \setminus \mathcal{H}' = \{H_i : i \in I \cup I' \cup J'\}$. If $i \in J'$, then by the induction hypothesis, $\iota(H_i, C') < (m_i + 1)/6$, so $6\iota(H_i, C') < m_i + 1$, and hence $6\iota(H_i, C') \leq m_i$. Thus,

$$J' = \{i \in [p] : \iota(H_i, C') \leq m_i/6\}. \quad (3)$$

We have $|E_G(v)| = d(v) = k \geq 3$. Let $A_1 \subseteq E_G(v)$ with $|A_1| = 3$. Thus, $A_1 = \{vx_1, vx_2, vx_3\}$ for some $x_1, x_2, x_3 \in N(v)$. Let $A_2 = \{x_H y_H : H \in \mathcal{H}\}$. Let $M_1 = E(G[N[v]])$, $M_2 = E_G(N(v), V(G'))$ and $M_3 = \bigcup_{H \in \mathcal{H}} E(H)$. Thus, $A_1 \subseteq M_1$, $A_2 \subseteq M_2$ and $m = |M_1| + |M_2| + |M_3|$. Let $a = |M_1 \setminus A_1| + |M_2 \setminus A_2|$ and $b = |\{x_{H_i} y_{H_i} : i \in J'\}|$. We have

$$\begin{aligned} m &= |A_1| + |A_2| + a + \sum_{i \in I \cup I' \cup J \cup J'} m_i \\ &= 3 + a + b + \sum_{i \in J'} m_i + \sum_{i \in I \cup I' \cup J} |E(H_i) \cup \{x_{H_i} y_{H_i}\}|. \end{aligned} \quad (4)$$

Case 1: $\mathcal{H}' = \emptyset$. Then, $J = \emptyset$. By Lemma 1 (with $X = \{v\}$ and $Y = N[v]$), Lemma 2 and (4),

$$\begin{aligned} \iota(G, C') &\leq 1 + \iota(G', C') = 1 + \sum_{H \in \mathcal{H}} \iota(H, C') \leq 1 + \sum_{i \in J'} \frac{m_i}{6} + \sum_{i \in I \cup I'} \frac{m_i + 1}{6} \\ &= 1 + \frac{m - 3 - a - b}{6} = \frac{m + 3 - a - b}{6}. \end{aligned}$$

If $a + b \geq 3$, then $\iota(G, C') \leq m/6$. Suppose $a + b \leq 2$.

Case 1.1: $G[N[v]]$ contains a C' -graph G_0 . We have $|V(G_0)| \geq 4$. If we assume that $|N[v]| \geq 5$, then we obtain $a \geq 3$, which contradicts $a + b \leq 2$. Thus, $N[v] = \{v, x_1, x_2, x_3\} = V(G_0)$. We may assume that $E(G_0) = \{vx_1, x_1x_2, x_2x_3, x_3v\}$. Since $a + b \leq 2$, we have $E(G) = A_1 \cup \{x_1x_2, x_2x_3\} \cup A_2 \cup M_3$ and $b = 0$. Since $b = 0$, $J' = \emptyset$. By Proposition 1, for any $i \in I \cup I'$, H_i has a C' -isolating set D_i with $y_{H_i} \in D_i$ and $|D_i| = (m_i + 1)/6$. Clearly, $\bigcup_{i \in I \cup I'} D_i$ is a C' -isolating set of G , so $\iota(G, C') \leq \sum_{i \in I \cup I'} |E(H_i) \cup \{x_{H_i} y_{H_i}\}|/6 = (m - 5)/6$.

Case 1.2: $G[N[v]]$ contains no C' -graph. Suppose $M_2 = A_2$. Then, $E_G(V(H), N[v]) = \{x_H y_H\}$ for each $H \in \mathcal{H}$. By Lemma 4 and Lemma 2,

$$\iota(G, C') \leq \iota(G', C') = \sum_{H \in \mathcal{H}} \iota(H, C') \leq \sum_{H \in \mathcal{H}} \frac{|E(H) \cup \{x_H y_H\}|}{6} = \frac{m - 3 - a}{6}.$$

Now suppose $M_2 \neq A_2$. Then, $a \geq 1$ and there exist $x \in N(v)$, $i \in [p]$ and $y \in V(H_i)$ such that $xy \in E(G)$ and $xy \neq x_{H_i}y_{H_i}$. Let $G^* = G - V(H_i)$. Clearly, G^* is connected and is not a 4-cycle (as $|N[v]| \geq 4$ and $G[N[v]] \not\cong C_4$). By the induction hypothesis,

$$\iota(G^*, \mathcal{C}') \leq \frac{(m - |E(H_i) \cup \{x_{H_i}y_{H_i}, xy\}|) + 1}{6} = \frac{m - m_i - 1}{6}.$$

Let D^* be a $\mathcal{C}'$ -isolating set of G^* of size $\iota(G^*, \mathcal{C}')$.

Case 1.2.1: $i \in I$. Let $X = \{w_{i,j}^1: j \in [q_i]\}$, $Y = \{w_{i,j}^3: j \in [q_i]\}$ and $Y' = V(H_i) \setminus Y$. Let $G_Y = G - Y'$. Suppose $d_{G_Y}(w) \leq 1$ for each $w \in Y$. Then, by Corollary 2, $\iota(G_Y, \mathcal{C}') = \iota(G_Y - Y, \mathcal{C}') = \iota(G^*, \mathcal{C}')$. Since $Y' \subseteq N[X]$, Lemma 1 gives us

$$\iota(G, \mathcal{C}') \leq |X| + \iota(G_Y, \mathcal{C}') = q_i + \iota(G^*, \mathcal{C}') \leq \frac{m_i + 1}{6} + \frac{m - m_i - 1}{6} = \frac{m}{6}.$$

Now suppose $d_{G_Y}(w) \geq 2$ for some $w \in Y$. We have $w = w_{i,j}^3$ for some $j \in [q_i]$. Since $w_{i,j}^2, w_{i,j}^4 \in N(w) \setminus V(G_Y)$, $d(w) \geq 4$. Since $d(v) = \Delta(G)$, $d(v) \geq 4$. Since $a + b \leq 2$, it follows that $M_1 \setminus A_1 = \{vx_4\}$ for some $x_4 \notin \{x_1, x_2, x_3\}$, $M_2 \setminus A_2 = \{xy\}$, $E_{G_Y}(w) = \{x_{H_i}y_{H_i}, xy\} = \{x_{H_i}w, xw\}$ and $E(G) = A_1 \cup A_2 \cup \{vx_4, xw\} \cup M_3$. Let $X' = \{w\} \cup (X \setminus \{w_{i,j}^1\})$ and $D = X' \cup D^*$. Since $E_G(V(G^*), V(H_i)) = E_G(w)$ and X' is a $\mathcal{C}'$ -isolating set of H_i , D is a $\mathcal{C}'$ -isolating set of G . Therefore, we have

$$\iota(G, \mathcal{C}') \leq |X'| + |D^*| = q_i + \iota(G^*, \mathcal{C}') \leq \frac{m_i + 1}{6} + \frac{m - m_i - 1}{6} = \frac{m}{6}.$$

Case 1.2.2: $i \in I'$. Then, $H_i \simeq C'_4$ or $H_i \simeq C_5$. Thus, there exists some $w \in V(H_i)$ such that $y_{H_i}, y \in N[w]$ (because if $y_{H_i} \neq y$, then y_{H_i} and y are of distance at most 2 in H_i). Let $Y = V(H_i) \setminus N_{H_i}[w]$, $Y' = N_{H_i}[w]$ and $G_Y = G - Y'$. Then, $x_{H_i}y_{H_i}, xy \notin E(G_Y)$. Since $a + b \leq 2$, $|E_{G_Y}(V(G^*), Y)| \leq 1$. Thus, for some $x^* \in V(G^*)$, $N_{G_Y}[Y] \cap V(G^*) \subseteq \{x^*\}$ and $G_Y[\{x^*\} \cup Y]$ contains no $\mathcal{C}'$ -graph. Since $G^* = G_Y - Y$, Lemma 3 yields $\iota(G_Y, \mathcal{C}') = \iota(G^*, \mathcal{C}')$. By Lemma 1,

$$\iota(G, \mathcal{C}') \leq 1 + \iota(G_Y, \mathcal{C}') = 1 + \iota(G^*, \mathcal{C}') \leq \frac{m_i + 1}{6} + \frac{m - m_i - 1}{6} = \frac{m}{6}.$$

Case 1.2.3: $i \in J'$. Then, $b \geq 1$. Since $a + b \leq 2$ and $a \geq 1$, we have $a = b = 1$, $J' = \{i\}$ and $E(G) = A_1 \cup A_2 \cup \{xy\} \cup M_3$. For each $j \in I'$, let $D_j = \{y_{H_j}\}$. By Proposition 1, for each $j \in I$, H_j has a $\mathcal{C}'$ -isolating set D_j with $y_{H_j} \in D_j$ and $|D_j| = q_j$. Let $X = \bigcup_{j \in [p] \setminus \{i\}} (N[D_j] \cap V(H_j))$ and $D_X = \bigcup_{j \in [p] \setminus \{i\}} D_j$. Let $G_v^* = G[N[v] \cup V(H_i)]$. Then, G_v^* is a component of $G - X$, and any other component of $G - X$ contains no $\mathcal{C}'$ -graph. Let D_v^* be a $\mathcal{C}'$ -isolating set of G_v^* of size $\iota(G_v^*, \mathcal{C}')$. Then, $D_v^* \cup D_X$ is a $\mathcal{C}'$ -isolating set of G . Let $x' \in \{x_1, x_2, x_3\} \setminus \{x_{H_i}, x\}$. Since $E_G(v) = A_1$ and $E_G(N(v), V(H_i)) = \{x_{H_i}y_{H_i}, xy\}$, x' is a leaf of G_v^* . By Corollary 1, $\iota(G_v^*, \mathcal{C}') = \iota(G_v^* - x', \mathcal{C}')$.

Suppose $G_v^* - x' \not\cong C_4$. By the induction hypothesis, $\iota(G_v^* - x', \mathcal{C}') \leq (|E(G_v^* - x')| + 1)/6$, so $\iota(G_v^*, \mathcal{C}') \leq |E(G_v^*)|/6$. We have

$$\iota(G, \mathcal{C}') \leq |D_v^*| + |D_X| = \iota(G_v^*, \mathcal{C}') + \sum_{j \in [p] \setminus \{i\}} |D_j|$$

$$\leq \frac{|E(G_v^*)|}{6} + \sum_{j \in [p] \setminus \{i\}} \frac{|E(H_j) \cup \{x_{H_j} y_{H_j}\}|}{6} = \frac{m}{6}.$$

Now suppose $G_v^* - x' \simeq C_4$. Then, $V(H_i) = \{y\} = \{y_{H_i}\}$, $x \neq x_{H_i}$ and $E(G_v^* - x') = \{vx_{H_i}, x_{H_i}y, yx, xv\}$. If $I \cup I' = \emptyset$, then $E(G) = E(G_v^* - x') \cup \{vx'\}$, so G is a pure $(5, C_4)$ -special graph. Suppose $I \cup I' \neq \emptyset$.

Suppose $I' \neq \emptyset$. Let $h \in I'$. Then, $(D_X \setminus D_h) \cup \{x_{H_h}\}$ is a $\mathcal{C}'$ -isolating set of G , so

$$\iota(G, \mathcal{C}') \leq |D_X| = \sum_{j \in [p] \setminus \{i\}} |D_j| = \sum_{j \in [p] \setminus \{i\}} \frac{|E(H_j) \cup \{x_{H_j} y_{H_j}\}|}{6} < \frac{m}{6}.$$

Now suppose $I' = \emptyset$. Then, $I \neq \emptyset$. Suppose $y_{H_h} \notin \{v_{h,j} : j \in [q_h]\}$ for some $h \in I$. Then, $y_{H_h} = w_{h,j'}^t$ for some $j' \in [q_h]$ and $t \in [4]$. Let $D'_h = \{x_{H_h}\} \cup \{v_{h,j} : j \in [q_h] \setminus \{j'\}\}$. Then, $(D_X \setminus D_h) \cup D'_h$ is a $\mathcal{C}'$ -isolating set of G , so $\iota(G, \mathcal{C}') < m/6$ as above. Now suppose $y_{H_h} \in \{v_{h,j} : j \in [q_h]\}$ for each $h \in I$. If $x_{H_h} \neq x'$ for some $h \in I$, then D_X is a $\mathcal{C}'$ -isolating set of G , so $\iota(G, \mathcal{C}') < m/6$ as above. Suppose $x_{H_h} = x'$ for each $h \in I$. Then, $G[D_X \cup \{x'\}]$ is a tree and G is a pure (m, C_4) -special graph whose C_4 -constituents are $G_{1,1}, \dots, G_{1,q_1}, \dots, G_{p,1}, \dots, G_{p,q_p}$ and $G[N[v] \cup \{y\}]$.

Case 2: $\mathcal{H}' \neq \emptyset$. Let $H' \in \mathcal{H}'$. Let $x \in N(v)$ such that H' is linked to x . Thus, H' is a 4-cycle $(\{y_1, y_2, y_3, y_4\}, \{y_1y_2, y_2y_3, y_3y_4, y_4y_1\})$ with $xy_1 \in E(G)$. Let $\mathcal{H}_1 = \{H \in \mathcal{H}' : H \text{ is linked to } x \text{ only}\}$ and $\mathcal{H}_2 = \{H \in \mathcal{H} \setminus \mathcal{H}' : H \text{ is linked to } x \text{ only}\}$.

Case 2.1: Each member of $\mathcal{H}'$ is linked to at least two members of $N(v)$. Then, $\mathcal{H}_1 = \emptyset$ and H' is linked to some $x' \in N(v) \setminus \{x\}$, so $x'y' \in E(G)$ for some $y' \in V(H')$. Let $Y = \{x, y_1, y_2, y_4\}$ and $G^* = G - Y$. Then, G^* has a component G_v^* with $N[v] \setminus \{x\} \subseteq V(G_v^*)$, and $\{G_v^*\} \cup \mathcal{H}_2 \subseteq C(G^*) \subseteq \{G_v^*, (\{y_3\}, \emptyset)\} \cup \mathcal{H}_2$. If $y_3 \notin V(G_v^*)$, then y_3 is an isolated vertex of G^* , so $\iota(G^*, \mathcal{C}') = \iota(G^* - y_3, \mathcal{C}')$ by Corollary 1. By Lemma 2, $\iota(G^*, \mathcal{C}') = \iota(G_v^*, \mathcal{C}') + \sum_{H \in \mathcal{H}_2} \iota(H, \mathcal{C}')$. Since $Y \subseteq N[y_1]$, Lemma 1 yields

$$\iota(G, \mathcal{C}') \leq 1 + \iota(G^*, \mathcal{C}') \leq \frac{|E(H') \cup \{xy_1, vx\}|}{6} + \iota(G_v^*, \mathcal{C}') + \sum_{H \in \mathcal{H}_2} \frac{|E(H) \cup \{xy_{x,H}\}|}{6}.$$

Suppose $G_v^* \not\simeq C_4$. By the induction hypothesis, $\iota(G_v^*, \mathcal{C}') \leq (|E(G_v^*)| + 1)/6$, so $\iota(G, \mathcal{C}') \leq (m + 1)/6$. Suppose $\iota(G, \mathcal{C}') = (m + 1)/6$. Then, $\iota(G, \mathcal{C}') = 1 + \iota(G^*, \mathcal{C}')$, $\iota(G_v^*, \mathcal{C}') = (|E(G_v^*)| + 1)/6$, $\iota(H, \mathcal{C}') = (|E(H)| + 1)/6$ for each $H \in \mathcal{H}_2$, and

$$E(G) = E(H') \cup \{xy_1, vx\} \cup E(G_v^*) \cup \bigcup_{H \in \mathcal{H}_2} (E(H) \cup \{xy_{x,H}\}). \quad (5)$$

By the induction hypothesis, each member F of $\{G_v^*\} \cup \mathcal{H}_2$ is a pure $(|E(F)|, C_4)$ -special graph or a $\{C'_4, C'_5\}$ -graph. By Proposition 1, G_v^* has a $\mathcal{C}'$ -isolating set D_v^* with $x' \in D_v^*$ and $|D_v^*| = \iota(G_v^*, \mathcal{C}')$, and for each $H \in \mathcal{H}_2$, H has a $\mathcal{C}'$ -isolating set D'_H with $y_{x,H} \in D'_H$ and $|D'_H| = \iota(H, \mathcal{C}')$. Let $D = D_v^* \cup \bigcup_{H \in \mathcal{H}_2} D'_H$. By (5), we have $x'y' \in E(G_v^*)$, so $y' = y_3 \in V(G_v^*)$. Also by (5), $E_G(V(G^*), Y) = \{vx, y_2y_3, y_3y_4\}$ and $E(G[Y]) = \{xy_1, y_1y_2, y_1y_4\}$. Let $H'' = G[Y]$ if $v \notin D$ and $\mathcal{H}_2 = \emptyset$, and let $H'' = G[Y] - x$ if $v \in D$

or $\mathcal{H}_2 \neq \emptyset$. Note that $x \in N[D]$ if and only if $v \in D$ or $\mathcal{H}_2 \neq \emptyset$. Since $v, y_3 \in N(x')$, $C(G - N[D]) = \{H''\} \cup C(G_v^* - N[D_v^*]) \cup \bigcup_{H \in \mathcal{H}_2} C(H - N[D_H'])$. Thus, we have that D is a $\mathcal{C}'$ -isolating set of G of size $\iota(G^*, \mathcal{C}')$, contradicting $\iota(G, \mathcal{C}') = 1 + \iota(G^*, \mathcal{C}')$.

Now suppose $G_v^* \simeq C_4$. Then, $\mathcal{H}' = \{H'\}$ and $d(v) = 3 = \Delta(G)$. We may assume that $x = x_1$ and $x' = x_2$. We have $G_v^* = (\{v, x_2, x_3, z\}, \{vx_2, x_2z, zx_3, x_3v\})$ for some $z \in V(G') \setminus Y$. Since $\Delta(G) = 3$, we obtain $z \notin V(H')$ (otherwise, $d(z) \geq |\{x_2, x_3\} \cup N_{H'}(z)| = 4$), $N(y_3) \setminus \{y_2, y_4\} \subseteq \{x_1\}$ (because if $N(y_3) \cap \{x_2, x_3\} \neq \emptyset$, then $y_3 \in V(G_v^*)$), $N(y_1) = \{x_1, y_2, y_4\}$ and $y' \in \{y_2, y_4\}$. We may assume that $y' = y_2$. Thus, $N(x_2) = \{v, z, y_2\}$, $N(y_2) = \{y_1, y_3, x_2\}$, $N(x_3) \setminus \{v, z\} \subseteq \{x_1, y_4\}$ and $N(y_4) \setminus \{y_1, y_3\} \subseteq \{x_1, x_3\}$. Let $X_1 = \{v, x_1, y_1\}$ and $F_1 = G[\{x_3, z, x_2, y_2, y_3, y_4\}]$. Then, $C(G - X_1) = \{F_1\} \cup \mathcal{H}_2$. Since $X_1 \subseteq N[x_1]$, $\iota(G, \mathcal{C}') \leq 1 + \iota(G - X_1, \mathcal{C}')$ by Lemma 1. If $x_3y_4 \notin E(G)$, then F_1 is a path, so $\bigcup_{H \in \mathcal{H}_2} D_H$ is a $\mathcal{C}'$ -isolating set of $G - X_1$, and hence we have

$$\iota(G, \mathcal{C}') \leq 1 + \sum_{H \in \mathcal{H}_2} \frac{|E(H)| + 1}{6} < \frac{|E(H') \cup E(G_v^*)|}{6} + \sum_{H \in \mathcal{H}_2} \frac{|E(H) \cup \{xy_{x,H}\}|}{6} < \frac{m}{6}.$$

Suppose $x_3y_4 \in E(G)$. Let $X_2 = \{v, x_1, x_3, z, y_1, y_2, y_4\}$. Then, $C(G - X_2) = \{(\{x_2\}, \emptyset), (\{y_3\}, \emptyset)\} \cup \mathcal{H}_2$, so $\bigcup_{H \in \mathcal{H}_2} D_H$ is a $\mathcal{C}'$ -isolating set of $G - X_2$. Since $X_2 = N[\{y_1, x_3\}]$, Lemma 1 yields $\iota(G, \mathcal{C}') \leq 2 + \iota(G - X_2, \mathcal{C}')$, so

$$\iota(G, \mathcal{C}') \leq \frac{|E(H') \cup E(G_v^*) \cup \{vx_1, x_1y_1, x_2y_2, x_3y_4\}|}{6} + \sum_{H \in \mathcal{H}_2} \frac{|E(H) \cup \{xy_{x,H}\}|}{6} \leq \frac{m}{6}.$$

Case 2.2: Some member of $\mathcal{H}'$ is linked to only one member of $N(v)$. Thus, we may assume that H' is linked to x only. Let $h_1 = |\mathcal{H}_1|$. Then, $h_1 \geq 1$ as $H' \in \mathcal{H}_1$. Let $X = \{x\} \cup \bigcup_{H \in \mathcal{H}_1} V(H)$ and $D_X = \{x\}$. Then, D_X is a $\mathcal{C}'$ -isolating set of $G[X]$, and

$$|D_X| = 1 \leq \frac{\sum_{H \in \mathcal{H}_1} |E(H) \cup \{xy_{x,H}\}| + 1}{6} = \frac{5h_1 + 1}{6}.$$

Let $G^* = G - X$. Then, G^* has a component G_v^* with $N[v] \setminus \{x\} \subseteq V(G_v^*)$, and $C(G^*) = \{G_v^*\} \cup \mathcal{H}_2$.

If $G_v^* \not\simeq C_4$, then by the induction hypothesis, G_v^* has a $\mathcal{C}'$ -isolating set D_v^* with $|D_v^*| = \iota(G_v^*, \mathcal{C}') \leq (|E(G_v^*)| + 1)/6$. If $G_v^* \simeq C_4$, then let $D_v^* = \{x\}$. Let $D = D_v^* \cup D_X \cup \bigcup_{H \in \mathcal{H}_2} D_H$. By the definition of $\mathcal{H}_1$ and $\mathcal{H}_2$, $C(G - x) = \{G_v^*\} \cup \mathcal{H}_1 \cup \mathcal{H}_2$. Thus, D is a $\mathcal{C}'$ -isolating set of G as $x \in D$, $v \in V(G_v^*) \cap N[x]$ and D_X is a $\mathcal{C}'$ -isolating set of $G[X]$. We have

$$\begin{aligned} m &\geq |E(G_v^*) \cup \{vx\}| + \sum_{H \in \mathcal{H}_1 \cup \mathcal{H}_2} |E(H) \cup \{xy_{x,H}\}| \\ &= |E(G_v^*)| + 1 + 5h_1 + \sum_{H \in \mathcal{H}_2} (|E(H)| + 1). \end{aligned} \tag{6}$$

Suppose $G_v^* \simeq C_4$. Then, $D = \{x\} \cup \bigcup_{H \in \mathcal{H}_2} D_H$ and, by (6),

$$m \geq 5(h_1 + 1) + \sum_{H \in \mathcal{H}_2} (|E(H)| + 1) \geq 10 + \sum_{H \in \mathcal{H}_2} (|E(H)| + 1)$$

as $h_1 \geq 1$. We have

$$\iota(G, \mathcal{C}') \leq |D| = 1 + \sum_{H \in \mathcal{H}_2} |D_H| < \frac{10}{6} + \sum_{H \in \mathcal{H}_2} \frac{|E(H)| + 1}{6} \leq \frac{m}{6}.$$

Now suppose $G_v^* \not\cong C_4$. We have

$$\begin{aligned} \iota(G, \mathcal{C}') \leq |D| &= |D_v^*| + |D_x| + \sum_{H \in \mathcal{H}_2} |D_H| \leq \frac{|E(G_v^*)| + 1}{6} + 1 + \sum_{H \in \mathcal{H}_2} \frac{|E(H)| + 1}{6} \\ &\leq \frac{|E(G_v^*)| + 1}{6} + \frac{5h_1 + 1}{6} + \sum_{H \in \mathcal{H}_2} \frac{|E(H)| + 1}{6} \leq \frac{m + 1}{6} \quad (\text{by (6)}). \end{aligned} \quad (7)$$

Suppose $\iota(G, \mathcal{C}') = (m + 1)/6$. Then, equality holds throughout in each of (6) and (7). Consequently, $\iota(G, \mathcal{C}') = |D|$, $h_1 = 1$, $\mathcal{H}_1 = \{H'\}$, $|D_v^*| = (|E(G_v^*)| + 1)/6$, $|D_H| = (|E(H)| + 1)/6$ for each $H \in \mathcal{H}_2$, and

$$E(G) = E(G_v^*) \cup \{vx\} \cup E(H') \cup \{xy_{x,H'}\} \cup \bigcup_{H \in \mathcal{H}_2} (E(H) \cup \{xy_{x,H}\}).$$

Let $\mathcal{I} = \{F \in \{G_v^*\} \cup \mathcal{H}_2 : F \text{ is a pure } (|E(F)|, C_4)\text{-special graph}\}$ and $\mathcal{I}' = \{F \in \{G_v^*\} \cup \mathcal{H}_2 : F \text{ is a } \{C_4', C_5\}\text{-graph}\}$. By the induction hypothesis, $\{G_v^*\} \cup \mathcal{H}_2 = \mathcal{I} \cup \mathcal{I}'$.

Suppose $\mathcal{I}' \neq \emptyset$. Let $F' \in \mathcal{I}'$. If $F' = G_v^*$, then let $D' = D \setminus D_v^*$. If $F' \in \mathcal{H}_2$, then let $D' = D \setminus D_{F'}$. Since $x \in D'$, D' is a $\mathcal{C}'$ -isolating set of G . We have $\iota(G, \mathcal{C}') \leq |D| - 1$, a contradiction. Therefore, $\mathcal{I}' = \emptyset$, and hence $\mathcal{I} = \{G_v^*\} \cup \mathcal{H}_2$.

Let $F_1, \dots, F_r$ be the members of $\mathcal{I}$, where $F_1 = G_v^*$. Let $y_{x,F_1} = v$ and $D_{F_1} = D_v^*$. For each $i \in [r]$, let $F_{i,1}, \dots, F_{i,s_i}$ be the C_4 -constituents of F_i , and for each $j \in [s_i]$, let $u_{i,j}^0$ be the C_4 -connection of $F_{i,j}$, and let $u_{i,j}^1, \dots, u_{i,j}^4$ be the members of $V(F_{i,j}) \setminus \{u_{i,j}^0\}$ such that $E(F_{i,j}) = \{u_{i,j}^0 u_{i,j}^1, u_{i,j}^1 u_{i,j}^2, u_{i,j}^2 u_{i,j}^3, u_{i,j}^3 u_{i,j}^4, u_{i,j}^4 u_{i,j}^1\}$. Suppose $y_{x,F_h} \notin \{u_{h,j}^0 : j \in [s_h]\}$ for some $h \in [r]$. Then, $y_{x,F_h} = u_{h,j'}^t$ for some $j' \in [s_h]$ and $t \in [4]$. Since $x \in D$, $D \setminus \{y_{x,F_h}\}$ is a $\mathcal{C}'$ -isolating set of G . We have $\iota(G, \mathcal{C}') \leq |D| - 1$, a contradiction. Thus, $y_{x,F_h} \in \{u_{h,j}^0 : j \in [s_h]\}$ for each $h \in [r]$. Therefore, $G[\{u_{h,j}^0 : h \in [r], j \in [s_h]\} \cup \{x\}]$ is a tree, and G is a pure (m, C_4) -special graph whose C_4 -constituents are $F_{1,1}, \dots, F_{1,s_1}, \dots, F_{r,1}, \dots, F_{r,s_r}$ and $G[V(H') \cup \{x\}]$. $\square$

References

- [1] K. Bartolo, P. Borg, D. Scicluna, Isolation of squares in graphs, Discrete Mathematics, 347 (2024), paper 114161.
- [2] P. Borg, Isolation of cycles, Graphs and Combinatorics 36 (2020), 631–637.
- [3] P. Borg, Isolation of regular graphs and k -chromatic graphs, Mediterranean Journal of Mathematics 21 (2024), paper 148.
- [4] P. Borg, Proof of a conjecture on isolation of graphs dominated by a vertex, Discrete Applied Mathematics 371 (2025), 247–253.

- [5] P. Borg, Isolation of regular graphs, stars and k -chromatic graphs, *Discrete Mathematics* 349 (2026), paper 114706.
- [6] P. Borg, K. Fenech, P. Kaemawichanurat, Isolation of k -cliques, *Discrete Mathematics* 343 (2020), paper 111879.
- [7] P. Borg, K. Fenech, P. Kaemawichanurat, Isolation of k -cliques II, *Discrete Mathematics* 345 (2022), paper 112641.
- [8] R. Boutrig, M. Chellali, T.W. Haynes, S.T. Hedetniemi, Vertex-edge domination in graphs, *Aequationes Mathematicae* 90 (2016), 355–366.
- [9] G. Boyer, W. Goddard, Disjoint isolating sets and graphs with maximum isolation number, *Discrete Applied Mathematics* 356 (2024), 110–116.
- [10] Y. Caro, A. Hansberg, Partial domination - the isolation number of a graph, *Filomat* 31 (2017), 3925–3944.
- [11] S. Chen, Q. Cui, J. Zhang, A characterization of graphs with maximum cycle isolation number, *Discrete Applied Mathematics* 366 (2025), 161–175.
- [12] S. Chen, Q. Cui, L. Zhong, A characterization of graphs with maximum k -clique isolation number, *Discrete Mathematics* 348 (2025), 114531.
- [13] E.J. Cockayne, Domination of undirected graphs – A survey, in: *Lecture Notes in Mathematics*, Volume 642, Springer, 1978, 141–147.
- [14] E.J. Cockayne, S.T. Hedetniemi, Towards a theory of domination in graphs, *Networks* 7 (1977), 247–261.
- [15] Q. Cui, J. Zhang, A sharp upper bound on the cycle isolation number of graphs, *Graphs and Combinatorics* 39 (2023), paper 117.
- [16] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.
- [17] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Editors), *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc., New York, 1998.
- [18] S.T. Hedetniemi, R.C. Laskar (Editors), *Topics on Domination*, in: *Annals of Discrete Mathematics*, Volume 48, North-Holland Publishing Co., Amsterdam, 1991, Reprint of *Discrete Mathematics* 86 (1990).
- [19] S.T. Hedetniemi, R.C. Laskar, Bibliography on domination in graphs and some basic definitions of domination parameters, *Discrete Mathematics* 86 (1990), 257–277.
- [20] M. Lemańska, M. Mora, M.J. Souto-Salorio, Graphs with isolation number equal to one third of the order, *Discrete Mathematics* 347 (2024), paper 113903.

- [21] O. Ore, Theory of graphs, American Mathematical Society Colloquium Publications, Volume 38, American Mathematical Society, Providence, R.I., 1962.
- [22] X. Wei, G. Zhang, B. Zhao, On the C_4 -isolation number of a graph, arXiv:2310.17337 [math.CO].
- [23] D.B. West, Introduction to Graph Theory, second edition, Prentice Hall, 2001.
- [24] J. Yan, Isolation of the diamond graph, Bulletin of the Malaysian Mathematical Sciences Society 45 (2022), 1169–1181.
- [25] G. Zhang, B. Wu, A note on the cycle isolation number of graphs, Bulletin of the Malaysian Mathematical Sciences Society 47 (2024), paper 57.
- [26] G. Zhang, B. Wu, Cycle isolation of graphs with small girth, Graphs and Combinatorics 40 (2024), paper 38.
- [27] P. Żyliński, Vertex-edge domination in graphs, Aequationes Mathematicae 93 (2019), 735–742.