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APPROXIMATION BY NEURAL NETWORKS WITH A RESTRICTED SET OF WEIGHTS

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Abstract

Some approximation properties of the MLP (multilayer feedforward perceptron) model of neural networks have been investigated in a great deal of works over the last 30 years. It has been shown that for a large class of activation functions, a neural network can approximate arbitrarily well any given continuous function. The most significant result on this problem belongs to Leshno, Lin, Pinkus and Schocken. They proved that the necessary and sufficient condition for any single hidden layer network to have the u.a.p. (universal approximation property) is that its activation function not be a polynomial.

Some authors (White, Stinchcombe, Ito, and others) showed that a single hidden layer perceptron with some bounded weights can also have the u.a.p. Thus the weights required for u.a.p. are not necessary to be of an arbitrarily large magnitude. But what if they are too restricted? How can one learn approximation properties of networks with arbitrarily restricted set of weights? The current chapter makes a first step in solving this general problem. We consider neural networks with sets of weights consisting of two directions. Our purpose is to characterize all compact sets X in the n-dimensional space such that the network can approximate any continuous function over X.

Key Words: Neural network, MLP model, activation function, weight, path, orbit.

AMS Subject Classification: 41A30, 41A63, 68T05, 92B20.

1. Introduction

The theory of approximation of multivariate functions using artificial neural networks with one or more hidden layers is of great interest to both approximation theorists and applied mathematicians. At present, there are a large number of papers devoted to various problems in this area (see, e.g., [2-7, 10-14, 16, 17, 20]). We are interested in questions of density of a single hidden layer perceptron model in neural networks. A typical density result shows

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that a network can approximate an arbitrary function in a given class with any degree of accuracy.

A single hidden layer perceptron model with r units in the hidden layer and input $\mathbf{x} = (x_1, \ldots, x_n)$ evaluates a function of the form

$$\sum_{i=1}^{r} c_i \sigma(\mathbf{w}^i \cdot \mathbf{x} - \theta_i), \tag{1}$$

where the weights $\mathbf{w}^i$ are vectors in $\mathbb{R}^n$, the thresholds θ_i and the coefficients c_i are real numbers and the activation function σ is a univariate function which is considered to be continuous in the present note. For various activation functions σ, it has been proved in a number of papers that one can approximate well to a given continuous function from the set of functions of the form (1) (r is not fixed!) over any compact subset of $\mathbb{R}^n$. In other words, the set

$$\mathcal{M}(\sigma) = span\ \{\sigma(\mathbf{w} \cdot \mathbf{x} - \theta) : \ \theta \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^n\}$$

is dense in the space $C(\mathbb{R}^n)$ in the topology of uniform convergence on all compacta (see, e.g., [2,3,4,7,10]). More general result of this type belongs to Leshno, Lin, Pinkus and Schoken [11]. They proved that the necessary and sufficient condition for any continuous activation function to have the density property is that it not be a polynomial. This result shows the efficacy of the single hidden layer perceptron model within all possible choices of the activation function σ, provided that σ is continuous. In fact, density of the set $\mathcal{M}(\sigma)$ also holds for some reasonable sets of weights and thresholds. (see[17]).

Some authors showed that a single hidden layer perceptron with some restricted set of weights can also have the u. a. p. (universal approximation property). For example, White and Stinchcombe [20] proved that a single layer network with a polygonal, polynomial spline or analytic activation function and a bounded set of weights has the u.a.p. Ito [10] investigated this property of networks using monotone sigmoidal functions (tending to 0 at minus infinity and 1 at infinity), with only weights located on the unit sphere. We see that the weights required for u. a. p. are not necessary to be of an arbitrarily large magnitude. But what if they are too restricted. How can one learn approximation properties of networks with an arbitrarily restricted set of weights? This problem is too general to be solved directly in this form. But there are some cases which deserve a special attention. The most interesting case is, of course, neural networks with discrete sets of weights. To the best of our knowledge, approximation capabilities of such nets have not been studied yet. To be more precise, let W be a set of weights consisting of a finite number of vectors in $\mathbb{R}^n$. It is clear that if w varies only in W, the set $\mathcal{M}(\sigma)$ can not be dense in the topology of uniform convergence on all compacta. The problem here is in the determination of boundaries of efficacy of the model. Over which compact sets $X \subset \mathbb{R}^n$ does the model preserve its general propensity to approximate arbitrarily well every continuous multivariate function? In Section 2, we will answer this question for a set W of weights consisting of two vectors.

Clearly, well approximation by neural networks with weights varying only on two directions is not always possible. If such networks cannot approximate a prescribed multivariate function with arbitrarily small degree of accuracy, one may be interested in the error of approximation. In Section 3, we will give an explicit lower bound for the approximation error and find means of deciding if a given network is a best approximation.

2. Density Results

Before formulating our theorems, we recall which objects are called paths with respect to two directions $\mathbf{a}^1$ and $\mathbf{a}^2$ (see [1,8,9]). A path with respect to the directions $\mathbf{a}^1$ and $\mathbf{a}^2$, or simply a path if there is no confusion, is a finite or infinite ordered set of points $(\mathbf{x}^1, \mathbf{x}^2, \dots)$ in $\mathbb{R}^n$ with $\mathbf{x}^i \neq \mathbf{x}^{i+1}$ and its units $\mathbf{x}^{i+1} - \mathbf{x}^i$ alternatively perpendicular to the directions $\mathbf{a}^1$ and $\mathbf{a}^2$. The length of a path is the number of its points. A singleton is a path of the unit length. A path $\left(\mathbf{x}^1, \dots, \mathbf{x}^m\right)$ is closed if m is an even number and the set $\left(\mathbf{x}^1, \dots, \mathbf{x}^m, \mathbf{x}^1\right)$ also forms a path.

The relation $\mathbf{x} \sim \mathbf{y}$ when $\mathbf{x}$ and $\mathbf{y}$ belong to some path in a given compact set $X \subset \mathbb{R}^n$ defines an equivalence relation. The equivalence classes we call orbits.

Let K be a family of functions defined on $\mathbb{R}^n$ and X be a subset of $\mathbb{R}^n$. By K_X we will denote the restriction of this family to X.

We start the analysis by defining *ridge functions*. A *ridge function* is a multivariate function of the form

$$g\left(\mathbf{a} \cdot \mathbf{x}\right) = g\left(a_1 x_1 + \cdots + a_n x_n\right),$$

where $g : \mathbb{R} \to \mathbb{R}$ and $\mathbf{a} = (a_1, \dots, a_n)$ is a fixed vector (direction) in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. In other words, it is a multivariate function constant on the parallel hyperplanes $\mathbf{a} \cdot \mathbf{x} = \alpha, \alpha \in \mathbb{R}$. Ridge functions and their combinations arise in various contexts. They arise naturally in problems of partial differential equations (where they are called *plane waves*), computerized tomography, statistics, approximation theory, and neural networks (see e.g. [18] for further details).

Set

$$\mathcal{R}\left(\mathbf{a}^1, \mathbf{a}^2\right) = \left\{g_1\left(\mathbf{a}^1 \cdot \mathbf{x}\right) + g_2\left(\mathbf{a}^2 \cdot \mathbf{x}\right) : \ g_i \in C(\mathbb{R}),\ i = 1, 2\right\}.$$

The following theorem is a special case of the known general result of Marshall and O'Farrell [15] established for the sum of two algebras.

Theorem 1. *Let X be a compact subset of $\mathbb{R}^n$ with all its orbits closed. Then the set $\mathcal{R}_X\left(\mathbf{a}^1, \mathbf{a}^2\right)$ is dense in $C(X)$ if and only if X contains no closed path.*

Proof. Necessity. If X has closed paths, then X has closed paths $p' = (\mathbf{p}'_1, \dots, \mathbf{p}'_{2m})$ such that all points $\mathbf{p}'_1, \dots, \mathbf{p}'_{2m}$ are distinct. In fact, such special paths can be obtained from any closed path $p = (\mathbf{p}_1, \dots, \mathbf{p}_{2n})$ by the following simple algorithm: if the points of the path p are not all distinct, let i and $k > 0$ be the minimal indices such that $\mathbf{p}_i = \mathbf{p}_{i+2k}$; delete from p the subsequence $\mathbf{p}_{i+1}, \dots, \mathbf{p}_{i+2k}$ and call p the obtained path; repeat the above step until all points of p are all distinct; set $p' := p$. By Urison's great lemma, there exist continuous functions $h = h(\mathbf{x})$ on X such that $h(\mathbf{p}'_i) = 1$, $i = 1, 3, \dots, 2m-1$, $h(\mathbf{p}'_i) = -1$, $i = 2, 4, \dots, 2m$ and $-1 < h(\mathbf{x}) < 1$ elsewhere. Consider the measure

$$\mu_{p'} = \frac{1}{2m} \sum_{i=1}^{2m} (-1)^{i-1} \delta_{\mathbf{p}'_i},$$

where $\delta_{\mathbf{p}'_i}$ is a point mass at $\mathbf{p}'_i$. For this measure, $\int_X h \, d\mu_{p'} = 1$ and $\int_X g \, d\mu_{p'} = 0$ for all functions $g \in \mathcal{R}_X\left(\mathbf{a}^1, \mathbf{a}^2\right)$. Thus the set $\mathcal{R}_X\left(\mathbf{a}^1, \mathbf{a}^2\right)$ cannot be dense in $C(X)$.

Sufficiency. We are going to prove that the only annihilating regular Borel measure for $\mathcal{R}_X\left(\mathbf{a}^1,\mathbf{a}^2\right)$ is the zero measure. Suppose, contrary to this assumption, there exists a nonzero annihilating measure on X for $\mathcal{R}_X\left(\mathbf{a}^1,\mathbf{a}^2\right)$. The class of such measures with total variation not more than 1 we denote by S. Clearly, S is weak-* compact and convex. By the Krein-Milman theorem, there exists an extreme measure μ in S. Since the orbits are closed, μ must be supported on a single orbit. Denote this orbit by T.

For $i=1,2$, let X_i be the quotient space of X obtained by identifying the points $\mathbf{y}$ and $\mathbf{z}$ whenever $\mathbf{a}^i{\cdot}\mathbf{y}=\mathbf{a}^i{\cdot}\mathbf{z}$. Let π_i be the natural projection of X onto X_i. For a fixed point $t\in X$ set $T_1=\{t\}$, $T_2=\pi_1^{-1}(\pi_1T_1)$, $T_3=\pi_2^{-1}(\pi_2T_2)$, $T_4=\pi_1^{-1}(\pi_1T_3)$, $\ldots$ Obviously, $T_1\subset T_2\subset T_3\subset\cdots$. Therefore, for some $k\in\mathbb{N}$, $|\mu|(T_{2k})>0$, where $|\mu|$ is a total variation measure of μ. Since μ is orthogonal to every continuous function of the form $g\left(\mathbf{a}^1{\cdot}\mathbf{x}\right)$, $\mu(T_{2k})=0$. From the Haar decomposition $\mu(T_{2k})=\mu^+(T_{2k})-\mu^-(T_{2k})$ it follows that $\mu^+(T_{2k})=\mu^-(T_{2k})>0$. Fix a Borel subset $S_0\subset T_{2k}$ such that $\mu^+(S_0)>0$ and $\mu^-(S_0)=0$. Since μ is orthogonal to every continuous function of the form $g\left(\mathbf{a}^2{\cdot}\mathbf{x}\right)$, $\mu(\pi_2^{-1}(\pi_2S_0))=0$. Therefore, one can chose a Borel set S_1 such that $S_1\subset\pi_2^{-1}(\pi_2S_0)\subset T_{2k+1}$, $S_1\cap S_0=\varnothing$, $\mu^+(S_1)=0$, $\mu^-(S_1)\geqslant\mu^+(S_0)$. By the same way one can chose a Borel set S_2 such that $S_2\subset\pi_1^{-1}(\pi_1S_1)\subset T_{2k+2}$, $S_2\cap S_1=\varnothing$, $\mu^-(S_2)=0$, $\mu^+(S_2)\geqslant\mu^-(S_1)$, and so on.

The sets $S_0,S_1,S_2,\ldots$ are pairwise disjoint. For otherwise, there would exist positive integers n and m, with $n<m$ and a path $(y_n,y_{n+1},\ldots,y_m)$ such that $y_i\in S_i$ for $i=n,\ldots,m$ and $y_m\in S_m\cap S_n$. But then there would exist paths $(z_1,z_2,\ldots,z_{n-1},y_n)$ and $(z_1,z_2',\ldots,z_{n-1}',y_m)$ with z_i and z_i' in T_i for $i=2,\ldots,n-1$. Hence, the set

$$\{z_1,z_2,\ldots,z_{n-1},y_n,y_{n+1},\ldots,y_m,z_{n-1}',\ldots,z_2',z_1\}$$

would contain a closed path. This would contradict our assumption on X.

Now, since the sets $S_0,S_1,S_2,\ldots$ are pairwise disjoint, and $|\mu|(S_i)\geqslant\mu^+(S_0)>0$ for each $i=1,2,\ldots$, it follows that the total variation of μ is infinite. This contradiction completes the proof. ☐

Now we are able to step forward from ridge function approximation to neural networks.

Theorem 2. *Let $\sigma\in C(\mathbb{R})\cap L_p(\mathbb{R})$, where $1\le p<\infty$, or σ be a continuous, bounded, nonconstant function, which has a limit at infinity (or minus infinity). Let $W=\{\mathbf{a}^1,\mathbf{a}^2\}\subset\mathbb{R}^n$ be the given set of weights and X be a compact subset of $\mathbb{R}^n$ with all its orbits closed. Then*

$$\mathcal{M}_X(\sigma;W,\mathbb{R})=span\ \{\sigma(\mathbf{w}\cdot\mathbf{x}-\theta):\ \mathbf{w}\in W,\ \theta\in\mathbb{R}\}$$

is dense in the space of all continuous functions over X if and only if X contains no closed path.

Proof. Sufficiency. Let X be a compact subset of $\mathbb{R}^n$ with all its orbits closed. Besides, let X contain no closed path. By theorem 1, the set $\mathcal{R}_X\left(\mathbf{a}^1,\mathbf{a}^2\right)$ is dense in $C(X)$. This means that for any positive real number ε there exist continuous univariate functions g_1 and g_2 such that

$$\left|f(\mathbf{x})-g_1\left(\mathbf{a}^1{\cdot}\mathbf{x}\right)-g_2\left(\mathbf{a}^2{\cdot}\mathbf{x}\right)\right|<\frac{\varepsilon}{3} \tag{2}$$

for all $\mathbf{x}\in X$. Since X is compact, the sets $Y_i=\{\mathbf{a}^i{\cdot}\mathbf{x}:\ \mathbf{x}\in X\}$, $i=1,2$, are also compacts. In 1947, Schwartz [19] proved that continuous and p-th degree Lebesgue integrable univariate functions or continuous, bounded, nonconstant functions having a limit at infinity (or minus infinity) are not mean-periodic. Note that a function $f \in C(\mathbb{R}^n)$ is called mean periodic if the set *span* $\{f(\mathbf{x}-\mathbf{b}):\ \mathbf{b} \in \mathbb{R}^n\}$ is not dense in $C(\mathbb{R}^n)$ in the topology of uniform convergence on compacta (see [19]). Thus, Schwartz proved that the set

$$span\ \{\sigma(y-\theta):\ \theta \in \mathbb{R}\}$$

is dense in $C(\mathbb{R})$ in the topology of uniform convergence. We learned about this result from Pinkus [17, page 162]. This density result means that for the given ε there exist numbers $c_{ij}, \theta_{ij} \in \mathbb{R}$, $i=1,2,\ j=1,\dots,m_i$ such that

$$\left| g_i(y) - \sum_{j=1}^{m_i} c_{ij}\sigma(y-\theta_{ij}) \right| < \frac{\varepsilon}{3} \tag{3}$$

for all $y \in Y_i,\ i=1,2$. From (2) and (3) we obtain that

$$\left\| f(\mathbf{x}) - \sum_{i=1}^{2}\sum_{j=1}^{m_i} c_{ij}\sigma(\mathbf{a}^i \cdot \mathbf{x} - \theta_{ij}) \right\|_{C(X)} < \varepsilon. \tag{4}$$

Hence $\overline{\mathcal{M}_X(\sigma; W, \mathbb{R})} = C(X)$.

Necessity. Let X be a compact subset of $\mathbb{R}^n$ with all its orbits closed and the set $\mathcal{M}_X(\sigma; W, \mathbb{R})$ be dense in $C(X)$. Then for an arbitrary positive real number ε, inequality (4) holds with some coefficients $c_{ij}, \theta_{ij},\ i=1,2,\ j=1,\dots,m_i$. Since for $i=1,2$, $\sum_{j=1}^{m_i} c_{ij}\sigma(\mathbf{a}^i \cdot \mathbf{x} - \theta_{ij})$ is a function of the form $g_i(\mathbf{a}^i \cdot \mathbf{x})$, the subspace $\mathcal{R}_X\left(\mathbf{a}^1, \mathbf{a}^2\right)$ is dense in $C(X)$. Then by theorem 1, the set X contains no closed path. $\square$

Remark 1. It can be shown that the necessity of the theorem is valid without any restrictions on orbits of X. Indeed if X contains a closed path, then it contains a closed path $p = (\mathbf{x}^1, \dots, \mathbf{x}^{2m})$ with different points. The functional $G_p = \sum_{i=1}^{2m}(-1)^{i-1} f(\mathbf{x}^i)$ belongs to the annihilator of the subspace $\mathcal{R}_X\left(\mathbf{a}^1, \mathbf{a}^2\right)$. There exist nontrivial continuous functions f_0 on X such that $G_p(f_0) \neq 0$ (take, for example, any continuous function f_0 taking values $+1$ at $\{\mathbf{x}^1, \mathbf{x}^3, \dots, \mathbf{x}^{2m-1}\}$, -1 at $\{\mathbf{x}^2, \mathbf{x}^4, \dots, \mathbf{x}^{2m}\}$ and $-1 < f_0(\mathbf{x}) < 1$ elsewhere). This shows that the subspace $\mathcal{R}_X\left(\mathbf{a}^1, \mathbf{a}^2\right)$ is not dense in $C(X)$. But in this case, the set $\mathcal{M}_X(\sigma; W, \mathbb{R})$ cannot be dense in $C(X)$. The obtained contradiction means that our assumption is not true and X contains no closed path.

Remark 2. The hypothesis of the theorem on orbits of X cannot simply omitted in the sufficiency. The following example due to Marshall and O'Farrell justifies our assertion. For the sake of simplicity, we restrict ourselves to $\mathbb{R}^2$. Let $\mathbf{a}^1 = (1;1)$, $\mathbf{a}^2 = (1;-1)$ and the set of weights $W = \{\mathbf{a}^1, \mathbf{a}^2\}$. The set X can be constructed as follows. Let X_1 be the union of the four line segments $[(-3;0),(-1;0)]$, $[(-1;2),(1;2)]$, $[(1;0),(3;0)]$ and $[(-1;-2),(1;-2)]$. Rotate one segment in X_1 90° about its center and remove the middle one-third from each line segment. The obtained set denote by X_2. By the same way, one can construct X_3, X_4, and so on. It is clear that the set X_i has 2^{i+1} line segments and every orbit in X_i is a closed path consisting of 2^{i+1} points, one in each line segment. Let X be a limit of the sets X_i,

$i = 1,2,\dots$. Note that every orbit of X is dense in X, hence not closed. Besides, there are no closed paths.

By S_i, $i = \overline{1,4}$, denote the closed discs with the unit radius and centered at the points $(-2;0)$, $(0;2)$, $(2;0)$ and $(0;-2)$ respectively. Let P be a parallelogram with sides parallel to the vectors $\mathbf{a}^1$, $\mathbf{a}^2$ and containing the disks S_i, $i = \overline{1,4}$ (hence all the sets $X_1, X_2, \dots$, and X). Consider a continuous function f_0 such that $f_0(\mathbf{x}) = 1$ for $\mathbf{x} \in (S_1 \cup S_3) \cap X$, $f_0(\mathbf{x}) = -1$ for $\mathbf{x} \in (S_2 \cup S_4) \cap X$, and $-1 < f_0(\mathbf{x}) < 1$ elsewhere. Let $p = (\mathbf{y}^1, \mathbf{y}^2, \dots)$ be any infinite path in X. Since the points $\mathbf{y}^i$, $i = 1,2,\dots$, are alternatively in the sets $(S_1 \cup S_3) \cap X$ and $(S_2 \cup S_4) \cap X$, the path p is an extremal path for f_0. We say that a finite or infinite path $(p_1, p_2, \dots)$ is extremal for some function $u(x) \in C(Q)$ if $u(p_i) = (-1)^i \|u\|$, $i = 1,2,\dots$ or $u(p_i) = (-1)^{i+1} \|u\|$, $i = 1,2,\dots$ (see definition 2.4 in [8]). Extremal paths are associated with the following theorem (see theorem 2.5 in [8]):

Let $Q \subset \mathbb{R}^d$ be a convex compact set with the property: for any path $q = (q_1, q_2, \dots, q_n) \subset Q$ there exist points $q_{n+1}, q_{n+2}, \dots, q_{n+s} \in Q$ such that $(q_1, q_2, \dots, q_{n+s})$ is a closed path and s is not more than some positive integer N_0 independent of q. Then a necessary and sufficient condition for a function $g_0 \in \mathcal{R}_Q(\mathbf{a},\mathbf{b})$ to be a best approximation to the given function $f(x) \in C(Q)$ is the existence of a closed or infinite path $l = (p_1, p_2, \dots)$ extremal for the function $f_1(x) = f(x) - g_0(x)$.

By this theorem,

$$E(f_0, P) = \inf_{g \in \mathcal{R}_Q(\mathbf{a}^1,\mathbf{a}^2)} \|f_0 - g\|_{C(P)} = \|f_0\|_{C(P)} = 1. \tag{5}$$

Note that X does not satisfy the hypothesis of this theorem as regards convexity. But in fact, (5) remains valid for the error of approximation to f_0 over the set X. To show this, put $p_k = (\mathbf{y}^1, \dots, \mathbf{y}^k)$ and consider the path functional

$$G_{p_k}(f) = \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} f(\mathbf{y}^i).$$

G_{p_k} is a continuous linear functional obeying the following obvious properties:

(1) $\|G_{p_k}\| = G_{p_k}(f_0) = 1$;

(2) $G_{p_k}(g_1 + g_2) \le \frac{2}{k}(\|g_1\| + \|g_2\|)$ for ridge functions $g_1 = g_1\left(\mathbf{a}^1 \cdot \mathbf{x}\right)$ and $g_2 = g_2\left(\mathbf{a}^2 \cdot \mathbf{x}\right)$.

By property (1), the sequence $\{G_{p_k}\}_{k=1}^{\infty}$ has a weak* cluster point. This point will be denoted by G. By property (2), $G \in \mathcal{R}_X\left(\mathbf{a}^1,\mathbf{a}^2\right)^{\perp}$. Therefore,

$$1 = G(f_0) = G(f_0 - g) \le \|f_0 - g\|_{C(X)} \quad \text{for any } g \in \mathcal{R}_X\left(\mathbf{a}^1,\mathbf{a}^2\right).$$

Taking inf over g in the right-hand side of the last inequality, we obtain that $1 \le E(f_0, X)$. Now since $E(f_0, X) \le E(f_0, P)$, it follows from (5) that $E(f_0, X) = 1$. Recall that $\mathcal{M}_X(\sigma; W, \mathbb{R}) \subset \mathcal{R}_X\left(\mathbf{a}^1,\mathbf{a}^2\right)$. Thus

$$\inf_{h \in \mathcal{M}_X(\sigma; W, \mathbb{R})} \|f - h\|_{C(X)} \ge 1.$$

The last inequality finally shows that $\overline{\mathcal{M}_X(\sigma; W, \mathbb{R})} \ne C(X)$.

Examples:

(a) Let $\mathbf{a}^1$ and $\mathbf{a}^2$ be two noncollinear vectors in $\mathbb{R}^2$. Let $B = B_1 \cdots B_k$ be a broken line with the sides B_iB_{i+1}, $i = 1,\dots,k-1$, alternatively perpendicular to $\mathbf{a}^1$ and $\mathbf{a}^2$. Besides, let B does not contain vertices of any parallelogram with sides perpendicular to these vectors. Then the set $\mathcal{M}_B(\sigma;W,\mathbb{R})$ is dense in $C(B)$.

(b) Let $\mathbf{a}^1$ and $\mathbf{a}^2$ be two noncollinear vectors in $\mathbb{R}^2$. If X is the union of two parallel line segments, not perpendicular to any of the vectors $\mathbf{a}^1$ and $\mathbf{a}^2$, then the set $\mathcal{M}_X(\sigma;W,\mathbb{R})$ is dense in $C(X)$.

(c) Let now $\mathbf{a}^1$ and $\mathbf{a}^2$ be two collinear vectors in $\mathbb{R}^2$. Note that any path consisting of two points is automatically closed. Thus the set $\mathcal{M}_X(\sigma;W,\mathbb{R})$ is dense in $C(X)$ if and only if X contains no path different from a singleton. A simple example is a line segment not perpendicular to the given direction.

(d) Let X be any compact set with interior points. Then theorem 1 fails, since any such set contains the vertices of some parallelogram with sides perpendicular to the given directions $\mathbf{a}^1$ and $\mathbf{a}^2$, that is a closed path.

Assume $\mathcal{M}_X(\sigma;W,\mathbb{R})$ is dense in $C(X)$. Is it necessarily closed? The following theorem may describe cases when it is not.

Theorem 3. *Let $\mathbf{a}^1$ and $\mathbf{a}^2$ be fixed vectors in $\mathbb{R}^n$, $W = \{k_1\mathbf{a}^1, k_2\mathbf{a}^2\colon\ k_1,k_2 \in \mathbb{R}\}$ and $\mathcal{M}_X(\sigma;W,\mathbb{R}) = C(X)$. Then X contains no closed path and the lengths of all paths in X are bounded by some positive integer.*

Proof. Let $\mathcal{M}_X(\sigma;W,\mathbb{R}) = C(X)$. *Then* $\mathcal{R}_1 + \mathcal{R}_2 = C(X)$, where

$$\mathcal{R}_i = \{g_i(\mathbf{a}^i{\cdot}\mathbf{x}) :\ g_i \in C(\mathbb{R})\}, \quad i = 1,2.$$

Consider the linear operator

$$A : \mathcal{R}_1 \times \mathcal{R}_2 \to C(X), \quad A[(g_1,g_2)] = g_1 + g_2,$$

where $g_1 \in \mathcal{R}_1, g_2 \in \mathcal{R}_2$. The norm on $\mathcal{R}_1 \times \mathcal{R}_2$ we define as

$$\|(g_1,g_2)\| = \|g_1\| + \|g_2\|.$$

It is obvious that the operator A is continuous with respect to this norm. Besides, since $C(X) = \mathcal{R}_1 + \mathcal{R}_2$, A is a surjection. Consider the conjugate operator

$$A^* : C(X)^* \to [\mathcal{R}_1 \times \mathcal{R}_2]^*, \quad A^*[G] = (G_1,G_2),$$

where the functionals G_1 and G_2 are defined as follows

$$G_1(g_1) = G(g_1), g_1 \in \mathcal{R}_1; \quad G_2(g_2) = G(g_2), g_2 \in \mathcal{R}_2.$$

An element (G_1,G_2) from $[\mathcal{R}_1 \times \mathcal{R}_2]^*$ has the norm

$$\|(G_1,G_2)\| = \max\{\|G_1\|,\|G_2\|\}. \tag{6}$$

Let now $p=(p_1,\dots,p_m)$ be a path with different points: $p_i\neq p_j$ for any $i\neq j$, $1\leq i,\ j\leq m$. We associate with p the following functional over $C(X)$

$$L[f]=\frac{1}{m}\sum_{i=1}^{m}(-1)^{i-1}f(p_i).$$

Since $|L(f)|\leq\|f\|$ and $|L(g)|=\|g\|$ for a continuous function $g(\mathbf{x})$ such that $g(p_i)=1$, for odd indices $i,\ g(p_j)=-1$, for even indices j and $-1<g(\mathbf{x})<1$ elsewhere, we obtain that $\|L\|=1$. Let $A^*[L]=(L_1,L_2)$. One can easily verify that

$$\|L_i\|\leq\frac{2}{m},\quad i=1,2.$$

Therefore, from (6) we obtain that

$$\|A^*[L]\|\leq\frac{2}{m}. \tag{7}$$

Since A is a surjection, there exists $\delta>0$ such that

$$\|A^*[G]\|\geq\delta\|G\|\quad\text{ for any functional } G\in C(X)^*$$

Hence

$$\|A^*[L]\|\geq\delta. \tag{8}$$

Now from (7) and (8) we conclude that

$$m\leq\frac{2}{\delta}.$$

This means that the length of any path with different points is not more than $\left[\frac{2}{\delta}\right]+1$.

Let now $p=(p_1,\dots,p_m)$ be a path with at least two coinciding points. Then we can form a closed path with different points. This may be done by the following way: let i and j be indices such that $p_i=\ p_j$ and $j-i$ takes its minimal value. Note that in this case all the points $p_i,p_{i+1},\dots,p_{j-1}$ are distinct. Now if $j-i$ is an even number, then the path $(p_i,p_{i+1},\dots,p_{j-1})$, and if $j-i$ is an odd number, then the path $(p_{i+1},\dots,p_{j-1})$ is a closed path with different points. It remains to show that X can not possess closed paths with different points. Indeed, if $q=(q_1,\dots,q_{2k})$ is a path of this type, then the functional L, associated with q, annihilates all functions from $\mathcal{R}_1+\mathcal{R}_2$. On the other hand, $L[f]=1$ for a continuous function f on X satisfying the conditions $f(t)=1$ if $t\in\{q_1,q_3,\dots,q_{2k-1}\}$; $f(t)=-1$ if $t\in\{q_2,q_4,\dots,q_{2k}\}$; $f(t)\in(-1;1)$ if $t\in X\backslash q$. This implies that $\mathcal{R}_1+\mathcal{R}_2\neq C(X)$. Since $\mathcal{M}_X(\sigma;W,\mathbb{R})\subseteq\mathcal{R}_1+\mathcal{R}_2$, we obtain that $\mathcal{M}_X(\sigma;W,\mathbb{R})\neq C(X)$ on the contrary to our assumption. $\square$

For example, let $\mathbf{a}^1=(1;-1)$, $\mathbf{a}^2=(1;1)$, $W=\{\mathbf{a}^1,\mathbf{a}^2\}$ and σ be any continuous, bounded and nonconstant function, which has a limit at infinity. Consider the set

$$\begin{aligned}X\ =\ &\Big\{\Big(2;\frac{2}{3}\Big),\Big(\frac{2}{3};\frac{2}{3}\Big),(0;0),(1;1),\Big(1+\frac{1}{2};1-\frac{1}{2}\Big),\Big(1+\frac{1}{2}+\frac{1}{4};1-\frac{1}{2}+\frac{1}{4}\Big),\\ &\Big(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8};1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\Big),\dots\Big\}.\end{aligned}$$

It is clear that X is a compact set with all its orbits closed. (In fact, there is only one orbit, which coincides with X). Hence, by theorem 2, $\overline{\mathcal{M}_X(\sigma;W,\mathbb{R})}=C(X)$. But by theorem 3, $\mathcal{M}_X(\sigma;W,\mathbb{R})\neq C(X)$. Therefore, the set $\mathcal{M}_X(\sigma;W,\mathbb{R})$ is not closed in $C(X)$.

3. Approximation Error and Extremal Networks

If well approximation by neural networks is not possible, one may be interested in the error of this approximation. Below for one special class of bivariate functions, we give an easily calculable lower bound for the error of approximation by neural networks with any continuous activation function and weights consisting of two directions.

Let σ be any continuous univariate function on the real line, $W = \{k\mathbf{a}, t\mathbf{b} : \ k, t \in \mathbb{R}\}$ is the set of weights, where $\mathbf{a}, \mathbf{b}$ are linearly independent vectors in $\mathbb{R}^2$. For a compact set Ω in $\mathbb{R}^2$, the error of approximation of a given function $f \in C(\Omega)$ with networks from $\mathcal{M}_\Omega(\sigma; W, \mathbb{R})$ is denoted by $E(f, \mathcal{M})$.That is,

$$E(f, \mathcal{M}) \overset{def}{=} \inf_{g \in \mathcal{M}_\Omega(\sigma; W, \mathbb{R})} \|f - g\|.$$

Theorem 4. *Let*

$$\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 : c_1 \leq \mathbf{a} \cdot \mathbf{x} \leq d_1, \ \ c_2 \leq \mathbf{b} \cdot \mathbf{x} \leq d_2\right\},$$

where $\mathbf{a} = (a_1, a_2)$ *and* $\mathbf{b} = (b_1, b_2)$ *are linearly independent vectors,* $c_1 < d_1$ *and* $c_2 < d_2$. *Let a function* $f(\mathbf{x}) \in C(\Omega)$ *have the continuous partial derivatives* $\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2^2}$ *and for any* $\mathbf{x} = (x_1, x_2) \in \Omega$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(a_1 b_2 + a_2 b_1) - \frac{\partial^2 f}{\partial x_1^2} a_2 b_2 - \frac{\partial^2 f}{\partial x_2^2} a_1 b_1 \geq 0.$$

Then

$$E(f, \mathcal{M}) \geq \frac{1}{4}\left(f_1(c_1, c_2) + f_1(d_1, d_2) - f_1(c_1, d_2) - f_1(d_1, c_2)\right),$$

where

$$f_1(y_1, y_2) = f\left(\frac{y_1 b_2 - y_2 a_2}{a_1 b_2 - a_2 b_1}, \frac{y_2 a_1 - y_1 b_1}{a_1 b_2 - a_2 b_1}\right).$$

Proof. Introduce the new variables

$$y_1 = a_1 x_1 + a_2 x_2, \ \ y_2 = b_1 x_1 + b_2 x_2. \tag{9}$$

Since the vectors (a_1, a_2) and (b_1, b_2) are linearly independent, for any $(y_1, y_2) \in Y$, where $Y = [c_1, d_1] \times [c_2, d_2]$, there exists only one solution $(x_1, x_2) \in \Omega$ of the system (9). The coordinates of this solution are

$$x_1 = \frac{y_1 b_2 - y_2 a_2}{a_1 b_2 - a_2 b_1}, \qquad x_2 = \frac{y_2 a_1 - y_1 b_1}{a_1 b_2 - a_2 b_1}. \tag{10}$$

The linear transformation (10) transforms the function $f(x_1, x_2)$ to the function $f_1(y_1, y_2)$. Consider the approximation of $f_1(y_1, y_2)$ from the set

$$\mathcal{Z} = \{z_1(y_1) + z_2(y_2) : z_i \in C(\mathbb{R}), \ i = 1, 2\}.$$

It is easy to see that

$$E(f,\mathcal{M}) \geq E(f_1,\mathcal{Z}). \tag{11}$$

With each rectangle $S=[u_1,v_1]\times[u_2,v_2]\subset Y$ we associate the functional

$$L(h,S)=\frac{1}{4}\left(h(u_1,u_2)+h(v_1,v_2)-h(u_1,v_2)-h(v_1,u_2)\right), \quad h\in C(Y).$$

This functional has the following obvious properties:
(i) $L(z,S)=0$ for any $z\in\mathcal{Z}$ and $S\subset Y$.

(ii) For any point $(y_1,y_2)\in Y$, $L(f_1,Y)=\sum_{i=1}^{4}L(f_1,S_i)$, where $S_1=[c_1,y_1]\times[c_2,y_2]$, $S_2=[y_1,d_1]\times[y_2,d_2]$, $S_3=[c_1,y_1]\times[y_2,d_2]$, $S_4=[y_1,d_1]\times[c_2,y_2]$.

By the conditions of the theorem, it is not difficult to verify that

$$\frac{\partial^2 f_1}{\partial y_1\partial y_2}\geq 0 \text{ for any } (y_1,y_2)\in Y.$$

Integrating both sides of the last inequality over arbitrary rectangle $S=[u_1,v_1]\times[u_2,v_2]\subset Y$, we obtain that

$$L(f_1,S)\geq 0. \tag{12}$$

Set the function

$$f_2(y_1,y_2)=L(f_1,S_1)+L(f_1,S_2)-L(f_1,S_3)-L(f_1,S_4).$$

It is not difficult to verify that the function f_1-f_2 belongs to $\mathcal{Z}$. Hence

$$E(f_1,\mathcal{Z})=E(f_2,\mathcal{Z}). \tag{13}$$

Calculate the norm $\|f_2\|$. From the property (ii), it follows that

$$f_2(y_1,y_2)=L(f_1,Y)-2(L(f_1,S_3)+L(f_1,S_4))$$

and

$$f_2(y_1,y_2)=2\left(L(f_1,S_1)+L(f_1,S_2)\right)-L(f_1,Y).$$

From the last equalities and (12), we obtain that

$$|f_2(y_1,y_2)|\leq L(f_1,Y), \text{ for any } (y_1,y_2)\in Y.$$

On the other hand, one can check that

$$f_2(c_1,c_2)=f_2(d_1,d_2)=L(f_1,Y) \tag{14}$$

and

$$f_2(c_1,d_2)=f_2(d_1,c_2)=-L(f_1,Y). \tag{15}$$

Therefore,

$$\|f_2\|=L(f_1,Y). \tag{16}$$

Note that the points $(c_1,c_2),(c_1,d_2),(d_1,d_2),(d_1,c_2)$ in the given order form a closed path with respect to the directions $(0;1)$ and $(1;0)$. We conclude from (14)-(16) that this path is extremal for f_2. It is not difficult to verify that $z_0=0$ is a best approximation to f_2. Hence

$$E\left(f_2,\mathcal{Z}\right)=L\left(f_1,Y\right). \tag{17}$$

Now from (11),(13) and (17) we finally conclude that

$$E\left(f,\mathcal{M}\right)\geq L(f_1,Y)=\frac{1}{4}\left(f_1(c_1,c_2)+f_1(d_1,d_2)-f_1(c_1,d_2)-f_1(d_1,c_2)\right).$$

The last inequality completes the proof. □

Let, for example, (a_1,a_2) and (b_1,b_2) be basic vectors $(1,0)$ and $(0,1)$ correspondingly. As a set Ω take the unit square $[0,1]^2$. Let σ be any continuous function on $[0,1]$ and $f_0(x_1,x_2)=(x_1-\frac{1}{2})(x_2-\frac{1}{2})$. The function f_0 satisfies all the conditions of theorem 4. The approximating set of networks $\mathcal{M}$ has members of the form

$$\sum_{i=1}^{n_1}c_i\sigma(k_ix_1-\theta_i)+\sum_{j=1}^{n_2}d_j\sigma(t_jx_2-\lambda_j), \tag{18}$$

where $c_i,d_j,\theta_i,\lambda_j$ are arbitrary real numbers, k_i and t_j are real numbers different from zero and n_1,n_2 are positive integers. Applying theorem 4, we obtain that the error of approximation $E(f_0,\mathcal{M})$ of the function f_0 by networks of the form (18) is not less than $\frac{1}{4}$. On the other hand note that $E(f_0,\mathcal{M})\leq\|f_0\|=\frac{1}{4}$. Thus, $E(f_0,\mathcal{M})=\frac{1}{4}$.

At the end we are going to find conditions for characterization of extremal networks with weights from two directions. Fix a function $\sigma\in C(\mathbb{R})$ and vectors $\mathbf{a}^1,\mathbf{a}^2\in\mathbb{R}^d\backslash\{\mathbf{0}\}$. Consider neural networks from the set $\mathcal{M}(\sigma;W,\mathbb{R})$, where $W=\{k_1\mathbf{a}^1,k_2\mathbf{a}^2\colon\ k_1,k_2\in\ \mathbb{R}\}$. Let $f(\mathbf{x})$ be a given continuous function on some compact subset Q of $\mathbb{R}^d$. We want to find sufficient conditions for a network $\Xi\in\mathcal{M}_Q(\sigma;W,\mathbb{R})$ to be an extremal element (or a best approximation) to f. In other words, we want to characterize networks $\Xi=\Xi(\mathbf{x})=\sum_{i=1}^{2}\sum_{j=1}^{m_i}c_{ij}\sigma(\mathbf{a}^i\cdot\mathbf{x}-\theta_{ij})$ such that

$$\|f-\Xi\|=\max_{\mathbf{x}\in Q}|f(\mathbf{x})-\Xi(\mathbf{x})|=E(f),$$

where

$$E(f)=E(f,\mathcal{M}_Q)\overset{def}{=}\inf_{g\in\mathcal{M}_Q(\sigma;W,\mathbb{R})}\|f-g\|$$

is the error in approximating from $\mathcal{M}_Q(\sigma;W,\mathbb{R})$.

Theorem 5. *Let $Q\subset\mathbb{R}^d$ be a compact set with the property: for any path $q=(\mathbf{q}_1,\mathbf{q}_2,...,\mathbf{q}_n)\subset Q$ there exist points $\mathbf{q}_{n+1},\mathbf{q}_{n+2},...,\mathbf{q}_{n+s}\in Q$ such that $(\mathbf{q}_1,\mathbf{q}_2,...,\mathbf{q}_{n+s})$ is a closed path and s is not more than some positive integer N_0 independent of q. Then a sufficient condition for a network $\Xi(\mathbf{x})\in\mathcal{M}_Q(\sigma;W,\mathbb{R})$ to be extremal to the given function $f(\mathbf{x})\in C(Q)$ is the existence of a closed or infinite path $l=(\mathbf{p}_1,\mathbf{p}_2,\dots)$ such that $f(\mathbf{p}_i)-\Xi(\mathbf{p}_i)=(-1)^i\|f-\Xi\|, i=1,2,\dots$ or $f(\mathbf{p}_i)-\Xi(\mathbf{p}_i)=(-1)^{i+1}\|f-\Xi\|, i=1,2,\dots$*

Proof. Let $f \in C(Q)$, $\Xi \in \mathcal{M}_Q(\sigma; W, \mathbb{R})$, $l = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{2n})$ be closed path in Q and $f(\mathbf{p}_i) - \Xi(\mathbf{p}_i) = (-1)^i \|f - \Xi\|, i = 1, 2, \dots$ or $f(\mathbf{p}_i) - \Xi(\mathbf{p}_i) = (-1)^{i+1} \|f - \Xi\|$, $i = 1, 2, \dots$.

Consider the functional

$$G_l(f) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^{k+1} f(\mathbf{p}_k).$$

Note that for any network $g \in \mathcal{M}_Q(\sigma; W, \mathbb{R})$, $G_l(g) = 0$. That is, the functional G_l belongs to the annihilator of the set $\mathcal{M}_Q(\sigma; W, \mathbb{R})$.

It can be easily verified that

$$|G_l(f)| = \|f - \Xi\|. \tag{19}$$

and

$$|G_l(f)| \leq E(f). \tag{20}$$

It follows from (19),(20) and the definition of $E(f)$ that Ξ is an extremal element.

Let now a path $l = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n, \dots)$ be infinite and $f(\mathbf{p}_i) - \Xi(\mathbf{p}_i) = (-1)^i \|f - \Xi\|, i = 1, 2, \dots$ or $f(\mathbf{p}_i) - \Xi(\mathbf{p}_i) = (-1)^{i+1} \|f - \Xi\|$, $i = 1, 2, \dots$. Without loss of generality we may assume that all the points $\mathbf{p}_i$ are distinct (in the other case, we could form a closed path and prove in a few lines as above that Ξ is an extremal element). Consider the sequence $l_n = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)$, $n = 1, 2, ...$, of finite paths. By the condition, for each l_n there exists a closed path $l_n^{m_n} = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n, \mathbf{q}_{n+1}, ..., \mathbf{q}_{n+m_n})$, where $m_n \leq N_0$. Then for any positive integer n,

$$\left|G_{l_n^{m_n}}(f)\right| = \left|G_{l_n^{m_n}}(f - \Xi)\right| \leq \frac{n\|f - \Xi\| + m_n\|f - \Xi\|}{n + m_n} = \|f - \Xi\| \tag{21}$$

and

$$\left|G_{l_n^{m_n}}(f)\right| \geq \frac{n\|f - \Xi\| - m_n\|f - \Xi\|}{n + m_n} = \frac{n - m_n}{n + m_n}\|f - \Xi\|. \tag{22}$$

It follows from (21) and (22) that

$$\sup_{l_n^{m_n}} \left|G_{l_n^{m_n}}(f)\right| = \|f - \Xi\|. \tag{23}$$

Note that

$$\sup_{p \subset Q} |G_p(f)| \leq E(f), \tag{24}$$

where the sup is taken over all closed paths of Q. Now we deduce from (23) and (24) that

$$\|f - \Xi\| \leq E(f).$$

Hence Ξ is an extremal element. □

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TORSION FIELDS, QUANTUM GEOMETRIES, BROWNIAN MOTIONS AND STATISTICAL THERMODYNAMICS

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Abstract

We develop the relation between space-time and state-space quantum geometries with torsion fields (the so-called Riemann-Cartan-Weyl (RCW) geometries), statistical thermodynamics -and particularly the second law of thermodynamics- and their associated Brownian motions. In this setting, the metric conjugate of the trace-torsion one-form is the drift vector field of the Brownian motions. Thus, in the present approach Brownian motions are -in distinction with Nelson's Stochastic Mechanics- space-time structures. We extend this to the state-space of non-relativistic quantum mechanics and discuss the relation between a non-canonical quantum RCW geometry in state-space associated with the gradient of the quantum-mechanical expectation value of a self-adjoint operator. A particular case is given by the generalized laplacian operator defined by a RCW geometry, which is the generator of the space-time Brownian motions. We discuss the reduction of the wave function in terms of a RCW quantum geometry in state-space. We characterize the Schroedinger equation in terms of the RCW geometries and Brownian motions, for systems under observation as well as those unobserved. Thus, in this work, the Schroedinger field is a torsion generating field. In this work the U and R processes -in the sense of R. Penrose- are associated to RCW geometries and their Brownian motions, the former to RCW space-time geometries and their associated Brownian motions, and the latter to their extension to the state-space of nonrelativistic quantum mechanics given by the projective Hilbert space. In this setting, the Schroedinger equation can be either linear or nonlinear. We discuss the problem of the many times variables and the relation with dissipative processes. We present as an additional example of RCW geometries and their Brownian motions, the dynamics of viscous fluids obeying the invariant Navier-Stokes equations. We introduce in the present setting an extension of R. Kiehn's approach to dynamical systems starting from the notion of the topological dimension of one-forms, to apply it to the trace-torsion one-form whose metric conjugate is the Brownian motion's drift vector-field and discuss the topological notion of turbulence. In our setting, whenever the metric is not rivial, the quantum potential is found to coincide (up to a conformal factor) with the metric scalar curvature. We discuss the possible relations between the present approach and the nonlocal universal correlations between dissipative systems, first found by Kozyrev, and subsequently in diverse geophysical, solar and ionospheric

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observations. We introduce statistical thermodynamics in relation to fluctuations of systems, and its relations with the RCW geometries. We analyze the relations between these geometries and the fluctuation-dissipation relations. In the case of linear torsion we obtain the Onsager relations. We prove a non-linear Boltzmann theorem. We find that the free energy of the non-equilibrium systems decreases but on the zeros of the torsion-drift, signalling the onset of syntropic processes which in the case of dimension four we relate to the topological torsion introduced by Kiehn and the structure of the singularities associated to it. We discuss the relation with a time-arrow determined by the final equilibrium state and its associated torsion geometry.

1. Introduction

In a series of articles [63] [65] (and references therein) we have presented a fusion between space-time structures and Brownian motions, in which a complementarity of the objects characterizing the Brownian motion, i.e. the noise tensor which produces a metric, and the drift vector field which describes the average velocity of the Brownian motion whenever this takes place in space-time. These space and time structures, which can be defined starting from flat Euclidean or Minkowski space-time, have in addition to a metric, a torsion tensor which is formed from the metric conjugate of the drift vector field, and the laplacian operator defined by this geometrical structure is the differential generator of the Brownian motions. Thus , in this equivalence, one can choose the Brownian motions as the original structures determining a space-time structure, or conversely, the space-time structures produce a Brownian motion process. Thus, in view that the space and time geometries can be seen as associated with an extension of the theory of gravitation which in fact was first explored in joint work by Einstein with Cartan [14], then the foundations for the gravitational field, at least those associated to this restricted case of torsion reduced to the trace, can be found in these Brownian motions. Thus, in this equivalence, lies a characterization of the Universe in which due to the self-similarity of Brownian motions with its associated fractal structures, and the infinite velocity propagation of diffusion processes, point to a phenomenology which is not the classical mechanical metaphor, but one in which interactions at a point are imparted in no time to the whole Universe, while an hologram picture of reality (which recalls the Bohm conception of implicit order [7]), appears as its natural expression of universal scales that have been gauged to produce the actual geometries and the associated Brownian motions. Indeed, these space-time geometrical structures can be introduced by the Einstein λ transformations on the tetrad fields [14], from which the usual Weyl scale transformations can be deduced, but contrarily to Weyl geometries, these structures have torsion and they are integrable [65, 79] in contrast with Weyl's theory [94]. We have called these connections as RCW structures (short for Riemann-Cartan-Weyl); see [63, 65] and references therein. We have shown that light waves satisfying the wave-propagation equation and the eikonal ray equations of geometrical optics, already in the case of Mikowski spacetime generate these geometries, and still that the singularities of the torsion are the loci for quantum jumps [76].We have characterized the spinor and twistor geometries in the case of quaternionic waves [74] and related it to the multivalued Matrix Logic that has for particular cases quantum, fuzzy and Boolean logics [75]. We have further related them to the existence of a universal map of the body's sensorium and vortical structures on the neurocortex, as well as vortical structures in Matrix Logic which exist at the basis of cognition associated to Brownian motions as the neurocortex correlates of spacetime Brownian motions produced by photons [76–78].

This description in terms of gauging the scale transformations, begs the question about how universal scales can be to be able to produce a Universe of diversity which gives place to the phenomenae we call life, the quantum mechanical scales and still the planetary and galactic scales?[1]

We can further enquire what is the relation between the aether and a Universe described in terms of this equivalence, in which due to the fact that torsion is a non-metric geometrical object describing a topological obstruction to triviality, i.e. the breaking of closure of infinitesimal parallelograms in the particular scale we are describing this equivalence, so that the presence of a flow is intuitively evoked by this geometry, and the aether. This may seem strange to most of the readers, since the Michelson-Morley experiments seemingly disproved the existence of a background fluid, which years later reappeared in quantum field theory in the guise of the vacuum fluctuations [2]

This negative result called for the fusion of space and time, in a single structure which we know as Minkowski space, which the founding fathers of modern physics found it abstruse at the beginning. The fact is that the Lorentz group does not depend on the existence or not of an aether, and it has been associated by V. Fock to particles as space-time structures associated with solutions of the eikonal equation for which in Minkowski space this equation is Lorentz-invariant; see [19]. Furthermore, if an aether would exist, the Lorentz transformations, in contrary to common belief, does not lose its place, because they become the set of transformations by which two arbitrary observers can agree in the existence of a lump of space-time associated to the solution of the eikonal equation. Furthermore, the velocity of light as a universal factor does not loose its place. What about General Relativity (GR), vis-a-vis the existence of background Brownian motions, where we recall that GR

[1]The answer to this may come from the research by Hartmutt Muller, currently called Global Scaling, starting from research by the Russian biologist, Cislenko. This researcher discovered that biological species sizes can be represented in a logarithmic scale where they appear concentrated in specific equally distanced intervals; see. Cislenko, Structure of Fauna And Flora With Regard to Body Size of Organisms (Lomonosov-University Moscow, 1980). Analysis of data of natural processes and structures on all scales, from the cosmological to the quantum, have shown a similar behavior. The scales in which the Universe appears to be related to a fractal structure on this logarithmic scale, and the void sets of this vacuum are nodal sets for a stationary wave that appears from a model of interacting classical particles, and associated with the creation or annhilitation of particles, or, more universally, of structures. Thus, one can apply as a general method Muller's findings, to the analysis with this fractal structure of a time series of experimental data of arbitrary phenomenae, and thus to be able to deduce the possibility of creation (or annihilation) of hitherto unkwown structures; see [49]. Remarkably, this resonant effects of the vacuum through the nodes, has allowed to produce transference of information, concretely, computer files between computers unconnected through internet or other tangible communication scheme; see [50]. In the context of our theory, the logarithmic scale is gauged and introduces the exact term of the trace-torsion one-form, whose metric conjugate is part of the drift of the underlying Brownian motion through which the teleportation may be produced. Thus, the universality of global scaling laws lead to gauge dependent Brownian motions of all systems.

[2]There is a strong controversy about the interpretations of the Michelson-Morley experiments. The experiments were repeated extensively by Miller who interpreted his results as yielding a positive result for the existance of the aether [47]. More contemporary experiments with different settings, may point out to the existence of the aether [2, 8]. Einstein did not come to terms with the abandonment of the aether, as can be testimonied in his Leyden lectures [15].

appeared to give a geometrical invariant extension of Minkowksi space-time precisely to account of the existence of massive objects as deformations of the flat space-time? Does the principle of general covariance looses its ground as a basic tenant for the universality of the laws of physics? In this regard the fact that the theory of Brownian motion cannot be formulated without a diffeomorphism invariant distinction between the first and second moments, i.e. between the drift and the noise tensor, for which it is indispensable to introduce the notion of a linear connection as proved by Rapoport; see [63, 65, 67, 68]. This gives further support, albeit from an unexpected quantum status, to the relevance of the general principle of covariance, but now stemming from a more fundamental non-differentiable fractal level [3].

In this article we shall treat the problem of non-relativistic quantum mechanics in terms of diffusion processes both in spacetime and the state-space of quantum mechanics. Thus, in this approach, it will appear that the Schroedinger field can be associated with a scale field producing a distortion in the vacuum, and introducing as well the associated Brownian motions. There have been numerous attempts to relate non-relativistic quantum mechanics to diffusion equations; the most notable of them is Stochastic Mechanics , due to Nelson [52]. Already Schroedinger proposed in 1930-32 that his equation should be related to the theory of Brownian motions, and proposed a scheme he was not able to achieve, the so-called interpolation problem which requires to describe the Brownian motion and the wave functions in terms of interpolating the initial and final densities in a given time-interval [88]. More recently Nagasawa presented a solution to this interpolation pr oblem and further elucidated that the Schroedinger equation is in fact a Boltzmann equation [51]. Neither Nagasawa nor Nelson presented these Brownian motions as spacetime structures, but rather as matter fields *on* the vacuum. While Nelson introduced artificially a forward and backward stochastic derivatives to be able to reproduce the Schroedinger equation as a formally time-symmetric equation, Nagasawa was able to solve the interpolation problem in terms of the forward diffusion process and its adjoint backward process, from which without resort to the ad-hoc constructions due to Nelson, he was able to prove that this was related to the Kolmogorov characterization of time-irreversibility of diffusion processes in terms of the non-exact terms of the drift, here related to the trace-torsion. In spite of the ad-hoc character of Nelson's approach, a similar approach to quantization in terms of an initial fractal structure of space-time and the introduction of Nelson's forward and backward stochastic derivatives, was developed by Nottale in his Scale Theory of Relativity [55]. Remarkably, his approach has promoted the Schroedinger equation as valid for large scale structures, and predicted the existence of exo-solar planets which were observationally verified to exist [56]. This may further support the idea that the RCW structures introduced in the vacuum by scale transformations, are valid, as any topological approach would be, independently of the scale in which the associated Brownian motions and equations of quantum mechanics are posited. Furthermore, Kiehn has proved that the Schroedinger equation in

[3]Einstein somewhat conceded to the criticism of the so called operationalists, as Bridgeman and Kretschmann, on downplaying the role of the Principle of General Covariance; see E. Kretschmann, *Ann.der Physik* **53**, 575 (1917), P.W. Bridgeman, *Natural of Physical Theory*, Princeton Univ. Press (1936); if it would not have been by the developments of the mathematical theory of Brownian motions, and still, the inception of gauge-theoretical geometrical methods in statistical and condensed matter physics at its very roots, this criticism of the geometrical approach, and further, of a topological approach, would have prevailed.

spatial 2D can be exactly transformed into the Navier-Stokes equation for a compressible fluid, if we further take the kinematical viscosity ν to be $\frac{\hbar}{m}$ with m the mass of the electron. As we proved in [65, 67] the Navier-Stokes equations share with the Schroedinger equation, that both have a RCW geometry at their basis; while in the Navier-Stokes equations the trace-torsion is $\frac{-1}{2\nu}u$ with u the time-dependent velocity one-form of the viscous fluid, in the Schroedinger equation, the trace-torsion one-form incorporates the logarithmic differential of the wave function -just like in Nottale's theory [55]- and further the electromagnetic potential terms of the trace-torsion. This correspondence between trace-torsion one-forms is what lies at the base of Kiehn's correspondance, with an important addendum: While in the approach of the Schroedinger equation the probability density is related to the Schroedinger scale factor (in incorporating the complex phase) and the Born formula turns out to be a formula and not an hypothesis, under the transformation to the Navier-Stokes equations it turns out that the probability density of non-relativistic quantum mechanics, is the entrosphy density of the fluid, i.e. the square of the vorticity, which thus plays a *geometrical* role that substitutes the probability density. Thus, in this approach, while there may be virtual paths sustaining the random behaviour of particles (as is the case also of the Navier-Stokes equations [63,67] and the interference such as in the two-slit experiments can be interpreted as a superposition of Brownian paths [51], the probability density has a purely geometrical fluid-dynamical meaning (the squared length of the vorticity vector field). We shall present the relation between what we can now call RCW quantum geometries, with the representation of the Schroedinger equation in the projective state-space of non-relativistic quantum mechanics, and further present the problem of the reduction of the wave function, as related to a non-canonical geometry in state-space. This quantum RCW geometry has a metric which is not the usual Fubini-Study metric, but is related to an extension of the classical symplectic geometry treatment of the Schroedinger function in state-space, to include the observation process in terms of a noise term and a trace-torsion drift given by (a modification of) the gradient of the Hamiltonian function corresponding to the symplectic formalism. This Hamiltonian function is none other that the quantum mechanical expectation function defined by the quantum Hamiltonian operator, or more specifically, it can be the Laplacian operator associated to the RCW geometry which has a correlate as a Brownian motion in space-time. Thus, if one incorporates the observation process into the theory, still RCW geometries will play an important role, since the Schroedinger symplectic vector field is the natural drift vector field in state-space whenever the noise coefficient is zero. We shall extend this the relations between RCW geometries, their Brownian motions and the Schroedinger equation, to the strong interactions in the framework of Hadronic Mechanics and the isoSchroedinger equation. Finally, we shall relate non-equilibrium statistical thermodynamics to the RCW geometries and further apply this to characterize the behaviour of entropy in terms of the RCW geometries, particularly of the zeros of the torsion.

2. Riemann-Cartan-Weyl Geometry of Diffusions

In this section we follow our articles in [63]. In this article M denotes a smooth connected compact orientable n-dimensional manifold (without boundary). While in our initial works, we took for M to be spacetime, there is no intrinsic reason for this limitation, in fact if can be an arbitrary configuration manifold and still a phase-space associated to a dynamical system. The paradigmatical example of the latter, is the projective space associated to a finite-dimensional Hilbert-space of a quantum mechanical system. We shall further provide M with a linear connection described by a covariant derivative operator ∇ which we assume to be compatible with a given metric g on M, i.e. $\nabla g = 0$. Given a coordinate chart (x^α) $(\alpha = 1,\ldots,n)$ of M, a system of functions on M (the Christoffel symbols of ∇) are defined by $\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma(x)^\alpha_{\beta\gamma} \frac{\partial}{\partial x^\alpha}$. The Christoffel coefficients of ∇ can be decomposed as:

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{1}{2} K^\alpha_{\beta\gamma}. \tag{1}$$

The first term in (12) stands for the metric Christoffel coefficients of the Levi-Civita connection ∇^g associated to g, i.e. $\left\{ \begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix} \right\} = \frac{1}{2}(\frac{\partial}{\partial x^\beta} g_{\nu\gamma} + \frac{\partial}{\partial x^\gamma} g_{\beta\nu} - \frac{\partial}{\partial x^\nu} g_{\beta\gamma}) g^{\alpha\nu}$, and

$$K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta}, \tag{2}$$

is the cotorsion tensor, with $S^\alpha_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^\kappa_{\nu\gamma}$, and $T^\alpha_{\beta\gamma} = (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})$ the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to ∇, i.e. the operator acting on smooth functions on M defined as

$$H(\nabla) := 1/2 \nabla^2 = 1/2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \tag{3}$$

A straightforward computation shows that $H(\nabla)$ only depends in the trace of the torsion tensor and g, since it is

$$H(\nabla) = 1/2 \triangle_g + \hat{Q}, \tag{4}$$

with $Q := Q_\beta dx^\beta = T^\nu_{\nu\beta} dx^\beta$ the trace-torsion one-form and where $\hat{Q}$ is the vector field associated to Q via g: $\hat{Q}(f) = g(Q, df)$, for any smooth function f defined on M. Finally, $\triangle_g$ is the Laplace-Beltrami operator of g: $\triangle_g f = \mathrm{div}_g\, \mathrm{grad} f$, $f \in C^\infty(M)$, with div_g the Riemannian divergence. Thus for any smooth function, we have $\triangle_g f = 1/[det(g)]^{\frac{1}{2}} g^{\alpha\beta} \frac{\partial}{\partial x^\beta}([det(g)]^{\frac{1}{2}} \frac{\partial}{\partial x^\alpha} f)$. Furthermore, the second term in (14), i.e. $\hat{Q}$ coincides with the Lie-derivative with respect to the vectorfield $\hat{Q}$: $L_{\hat{Q}} = i_{\hat{Q}} d + d i_{\hat{Q}}$, where $i_{\hat{Q}}$ is the interior product with respect to $\hat{Q}$: for arbitrary vectorfields $X_1,\ldots,X_{k-1}$ and ϕ a k-form defined on M, we have $(i_{\hat{Q}}\phi)(X_1,\ldots,X_{k-1}) = \phi(\hat{Q}, X_1,\ldots,X_{k-1})$. Then, for f a scalar field, $i_{\hat{Q}} f = 0$ and

$$L_{\hat{Q}} f = (i_{\hat{Q}} d + d i_{\hat{Q}}) f = i_{\hat{Q}} df = g(Q, df) = \hat{Q}(f). \tag{5}$$

Thus, our laplacian operator admits being written as

$$H_0(g, Q) = \frac{1}{2} \triangle_g + L_{\hat{Q}}. \tag{6}$$

Consider the family of zero-th order differential operators acting on smooth k-forms, i.e. differential forms of degree k $(k = 0,\ldots,n)$ defined on M:

$$H_k(g, Q) := 1/2 \triangle_k + L_{\hat{Q}}, \tag{7}$$

In the first summand of the r.h.s. of (7) we have the Hodge operator acting on k-forms:

$$\triangle_k = (d-\delta)^2 = -(d\delta + \delta d), \tag{8}$$

with d and δ the exterior differential and codifferential operators respectively, i.e. δ is the adjoint operator of d defined through the pairing of k-forms on M: $(\omega_1, \omega_2) := \int \otimes^k g^{-1}(\omega_1, \omega_2) vol_g$, for arbitrary k-forms ω_1, ω_2, where $vol_g(x) = det(g(x))^{\frac{1}{2}} dx$ is the volume density, g^{-1} denotes the induced metric on 1-forms and $\otimes^k g^{-1}$ the induced metric on k-forms . The last identity in eq. (7) follows from the fact that $d^2 = 0$ so that $\delta^2 = 0$. Since this operator when $k = 0$ coincides with the Laplace-Beltrami operator $\triangle_g$, we see that from the family defined in eq. (7) we retrieve for scalar fields ($k = 0$) the operator $H(\nabla)$ defined in (4). The Hodge laplacian can be further written expliciting the Weitzenbock metric curvature term, so that when dealing with $M = R^n$ provided with the Euclidean metric, $\triangle_k$ is the standard Euclidean laplacian acting on the components of a k-form defined on R^n ($0 \leq k \leq n$).

Therefore, assuming that g is non-degenerate, we have defined a one-to-one mapping

$$\nabla \mapsto H_k(g, Q) = 1/2 \triangle_k + L_{\hat{Q}}$$

between the space of g-compatible linear connections ∇ with Christoffel coefficients of the form

$$\Gamma^{\alpha}_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + \frac{2}{(n-1)} \left\{ \delta^{\alpha}_{\beta} \, Q_{\gamma} - g_{\beta\gamma} \, Q^{\alpha} \right\}, \quad n \neq 1 \tag{9}$$

and the space of elliptic second order differential operators on k-forms ($k = 0, \ldots, n$).

Remarkably enough, the full torsion does not appear in the Laplacian operator associated to the connection, only the trace-torsion one-form Q that gives rise through its metric conjugate, to the drift interaction term. But the torsion tensor has as irreducible decomposition the form

$$T^{\alpha}_{\beta\gamma} \quad = \quad \frac{2}{n-1} \delta^{\alpha}_{[\beta} Q_{\gamma]} + \frac{1}{n-1} \varepsilon^{n}_{\beta\gamma\delta} \hat{T}^{\delta} + \bar{T}^{\alpha}_{\beta\gamma}, \tag{10}$$

where

$$\hat{T}_{\beta} = \frac{1}{2} \varepsilon jins T^{ins} \tag{11}$$

is the pseudovector term and the completely skew-symmetric term, $\bar{T}^{m}_{\alpha\beta}$ which then satisfies

$$\bar{T}_{\alpha\beta\gamma} + \bar{T}_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} = 0, \tag{12}$$

where $\bar{T}_{\alpha\beta\gamma} = g_{\alpha\delta} \bar{T}^{\delta}_{\beta\gamma}$. This is the term that was introduced in the joint collaborator by Einstein and Cartan, without identification of the physical nature of the term [12], and later, retaken in the framework of the Poincaré -gauge theory of gravitation, as the spin-angular-density of elementary particles or macroscopic objects [80]. As we shall seen already, the pseudovector and completely skew-symmetric terms do not appear in the generalized laplacian, and a fortiori do not appear in the expression of the Brownian motions that generate the RCW geometries. Thus, angular momentum is not a geometrical object which generates the Brownian motions, only the metric through the noise term that generates the metric through the relation we shall see in the next section, and the drift vector field given by the metric-conjugate of the trace-torsion. In other terms, the probability law of the Brownian motions are determined only by the noise and the trace-torsion, so that the angular momentum density plays no fundamental role in this respect. Nevertheless, since the Brownian motions of tensors and ultimately of differential forms, is determined by the probability law and the Brownian motions of the scalar particles and this information is determined by the scalar laplacian, so the diffusion of the angular-momentum is determined by the diffusion of the scalar fields, and naturally we would like to study the diffusion of angular momentum along the paths of the scalar fields. Thus, when considering the Navier-Stokes equations for viscous fluids, where the drift vector field associated to the geometrical-stochastic characterization of these equations is minus the fluid's velocity one-form obeying the Navier-Stokes equations, the diffusion of the angular momentum of the fluid, i.e. of the vorticity two-form could be identically characterized in terms of the diffusion of the Navier-Stokes laplacian, as an operator acting on scalars associated to a RCW connection. In this case, the diffusion equation for angular momentum is the Navier-Stokes equations for the vorticity, derived by simply applying the exterior differential to the Navier-Stokes equations for the velocity. In fact we can introduce a non-static completely antisymmetric torsion starting from the RCW connections in a most natural form which implies that it can be taken as derived from it and therefore it will propagate along the paths of the scalar particles generated by it. Indeed, it simply amounts to introduce the duality operation given by the Hodge star operator defined by the metric g,

$$* : \mathrm{sec}(\Lambda^k T^* M) \to \mathrm{sec}(\Lambda^{n-k} T^* M), A_k \mapsto *A_k, \tag{13}$$

and further apply it to the trace-torsion one-form, i.e. we consider the seudo-three-form $*Q$. Thus, if $\hat{Q}$ denotes the drift vectorfield given by the g-conjugate of Q, then $*Q = i_{\hat{Q}}\mathrm{vol}_{\mathrm{g}}$; see page 362 in Frankel [22]. Thus we note that this duality depends on the choice of an orientation, and thus $*Q$ has a built-in chirality associated to it. While this pseudo-three-form does not appear in the RCW laplacian, it is not an additional element of the structure, since it is naturally derived from the RCW geometrical structure. As a final comment, the equations of motion for the skew-symmetric torsion thus introduced, have to be deduced from the equations for Q itself, but we shall not elaborate on this further in the present article.

3. Riemann-Cartan-Weyl Diffusions

In this section we shall present recall the correspondence between RCW connections defined by (9) and diffusion processes of scalar fields having $H(g,Q)$ as infinitesimal generators (i.e. for short, in the following). For this, we shall see this correspondence in the case of scalars. Thus, naturally we have called these processes as *RCW diffusion processes*. For the extensions to describe the diffusion processes of differential forms, see [63].

For the sake of generality, in the following we shall further assume that $Q = Q(\tau, x)$ is a time-dependent 1-form. The stochastic flow associated to the diffusion generated by

$H_0(g,Q)$ has for sample paths the continuous curves $\tau \mapsto x(\tau) \in M$ satisfying the Ito invariant non-degenerate s.d.e. (stochastic differential equation)

$$dx(\tau) = X(x(\tau))dW(\tau) + \hat{Q}(\tau, x(\tau))d\tau. \tag{14}$$

In this expression, $X : M \times R^m \to TM$ is such that $X(x) : R^m \to TM$ is linear for any $x \in M$, the noise tensor, so that we write $X(x) = (X_i^\alpha(x))$ $(1 \le \alpha \le n,\ 1 \le i \le m)$ which satisfies

$$X_i^\alpha X_i^\beta = g^{\alpha\beta}, \tag{15}$$

where $g = (g^{\alpha\beta})$ is the expression for the metric in covariant form, and $\{W(\tau), \tau \ge 0\}$ is a standard Wiener process on R^m. Now , it is important to remark that here m can be arbitrary, i.e. we can take noise tensors defined on different spaces, and obtain the space diffusion process. In regards to the equivalence between the stochastic and the geometric picture, this enhances the fact that there is a freedom in the stochastic picture, which if chosen as the originator of the equivalence, points out to a more fundamental basis of the stochastic description. This is satisfactory, since it is impossible to identify all the sources for noise, and in particular those coming from the vacuum, which we take as the source for the randomness.

Here τ denotes the time-evolution parameter of the diffusion (in a relativistic setting it should not be confused with the time variable; we shall discuss more this issue further below), and for simplicity we shall assume always that $\tau \ge 0$. Indeed, taking in account the rules of stochastic analysis for which $dW^\alpha(\tau)dW^\beta(\tau) = \delta^\alpha_\beta d\tau$ (the Kronecker tensor), $d\tau dW(\tau) = 0$ and $(d\tau)^2 = 0$, we find that if $f : R \times M \to R$ is a C^2 function on the M-variables and C^1 in the τ-variable, then a Taylor expansion yields

$$f(\tau, x(\tau)) = f(0, x(0)) + \left[\frac{\partial f}{\partial \tau} + H_0(g,Q)f\right](\tau, x(\tau))d\tau + \frac{\partial f}{\partial x^\alpha}(\tau, x(\tau))X_i^\alpha(x(\tau))dW^i(\tau)$$

and thus $\frac{\partial}{\partial \tau} + H_0(g,Q)$ is the infinitesimal generator of the diffusion represented by integrating the s.d.e. (14). Furthermore, this identity sets up the so-called martingale problem approach to the random integration of linear evolution equations for scalar fields, and for the integration of the Navier-Stokes equation [67].Note, that if we start with eq. (14), we can reconstruct the associated RCW connection by using eq.(15) and the fact that the trace-torsion is the g-conjugate of the drift, i.e., in simple words, by lowering indexes of $\hat{Q}$ to obtain Q.

3.1. The Time Variables

Since the Michelson-Morley experiment on the existence of an aether were interpreted as giving negative results with regard to its existence, the introduction of the observer's time variable to account for the Lorentz transformations in the same status of the space variables, was the scheme of development of physics thereafter. Thus the notion of spacetime was born, the Minkowski metric was introduced as its first example, and the geometrization of physics ensued in terms of Lorentzian manifolds, in great measure due to the dissatisfaction of Einstein with regards to Special Relativity. In spite that a Lorentz invariant Brownian motion has been recently constructed by Oron and Horwitz [59] -and further applied to the equivalence of the Maxwell and Dirac-Hestenes equation [65]- in terms of a modification of the Gaussian distribution which turns out to be is invariant by Lorentz transformation, the whole program of quantum mechanics from the point of view of Feynman path integrals and its applications to quantum field theory requires an Euclidean signature for spacetime. Also, the construction of Brownian motions starting from the stochastic differential equations introduces an Euclidean spacetime structure in contrast with the Lorentzian degenerate metrics of General Relativity. So, if we wish to relate the spacetime geometry to Brownian motions *and* quantum mechanics, we need an Euclidean metric. The receipt for this has been to take the analytical continuation in the observers time variable. Another way of handling this time variable which has to do with an Euclidean signature, is to work with the universal time variable initially proposed by Stuckelberg [89] which by the way was the parameter used in quantum field theory. This choice can be further substantiated from the divergence-free classical theory of the electron recently proposed by Gill, Zachary and Lindesay [25]. In this theory, we equate the Minkowski metric $(dt)^2-(dx)^2-(dy)^2-(dz)^2$, where t is the time of the observer, with $(d\tau)^2$, where τ is the time of the source, or still, we can write this in the equivalent Euclidean metric $(dt)^2=(d\tau)^2+(dx)^2+(dy)^2+(dz)^2$. If we write the Lorentz-invariant equations of electromagnetism in the new Euclidean variables (τ,x,y,z), then we get a mathematically equivalent set of equations for electromagnetism; these equations in particular apply to the non-exact terms of the trace-torsion Q, as we shall see in this article. But, from the point of view of physics, there is a transformation between a passive time registered by the observer to a different quality of process, which we call time, and is proper to the source. To start with, τ is a non-integrable parameter, i.e. it is path-dependent, and thus it has to do with non-conservative processes. Thus the equations of electromagnetism while being mathematically equivalent in the Euclidean and Minkowski space, in the former case they have an additional term which is dissipative (and describes the radiation reaction) appearing in the wave equations of the electric and magnetic terms; this longitudinal term is proportional to the inner product of the velocity with the acceleration. In this setting, a classical theory for the electron without divergences is achieved. It was further proved that for a closed system of particles, a global inertial frame and unique invariant global time parameter for all observers is defined in [27, 89]. Thus τ which is the time-evolution parameter of the diffusion process (and in the general space and time manifold M case is not to be confused with the time variable t of General Relativity [4] may be related with the time variable introduced by Stueckelberg (and then introduced in quantum field theory), further elaborated by Piron and Horwitz and in several works by Horwitz and coworkers [27], Fanchi [17], Trump and Schieve [91], and in the context of a Schroedinger spacetime operator, by Kyprianidis [40], Collins and Fanchi [10] and Rapoport [63, 67, 71] [5]. Thus, the modification from the passive observer's time to the Euclidean time of the source allows to define simultaneity, while from the physical point of view, it has the meaning of a dissipative process being ascribed to the source. So, we

[4]David Bohm proposed in a paper that appeared in www.duversity, and presently inaccesible, that time was three-fold: time of the source, time of the observer and time of repetition, which he called hypparxis.

[5]Furthermore, the relativistic theory with the τ parameter predicts the interference in time of the wave function (see Horwitz and Rabin [28] which has been recently been verified experimentally [45]. We shall discuss further below a serious of experiments carried out by Koryzev and followers, where time appear as having an active role.

are very far from the trivial passive linear time variable which was incorporated by the Minkowski metric substituted here by a non-integrable function which allows to establish the universality of the observer itself. The fact that the evidence of the time variable which is no longer a mere registration by the observer, is the dissipative process associated to this transformation from the Minkowski space to the Euclidean one is remarkable. If one would downplay the sheer subsistence of Special Relativity with regards to the existence of the aether, if proven to exist, the role of geometries to describe physical processes is enhanced precisely if the Brownian motions described above are the very essence of this aether.

There has been in the last fifty years a number of experiments, mostly carried out in the former Soviet Union by Kozyrev, that have shown the existance of another role for time that the mere relational linear variable that we have inherited from Newtonian mechanics, and that in Special Relativity has been incorporated to the Minkowski metric. In these experiments the role of time appears precisely in terms of dissipative processes and it is evidenced through a field which cannot be shielded and propagates with an estimated velocity of $10^9 c$ [38]. Kozyrev interpreted his experiments as a proof of the reality of Minkowski space [39]. From the so called causal mechanics due to Kozyrev [37], Levich [44] and M.M. Lavrenteiev [43], it follows that asymmetrical (irreversible) time is an active substance, through which the transaction of distant dissipative processes of any nature can take place, being this transaction not only universal in nature, but also running both with retardation and advancement. The proposal of this formative character of time was forwarded not from an abstract quest, but from the need for solving astronomical and astrophysical problems. Kozyrev rejected the idea that the source for the stars energy were fusion reactions [13] [6], and proposed instead that a substantial active time was related to this [38]. In fact, recent measurements of the Sun, seem to confirm Koryzev's rejection to the present theory [48]. In this regard, the transactional interpretation of quantum mechanics was proposed as a possible explanation, and as a second perspective, the existence of nonlocal correlations in the strong macroscopic limit. This was applied to the forecast of geomagnetic and solar processes, with very good approximation with the actual processes that came to being after 12 3 days of observations [36]. In Kozyrev's theory, the active time parameter is realized through angular momentum, and thus it can be naturally be associated with a completely skewsymmetric torsion tensor. [7] In the presentation of the relation between the Schroedinger equation, torsion fields and Brownian motions, we shall see that the actual irreversible time invariant Brownian motion of the process that can be linked with the RCW connection with trace-torsion given by electromagnetic potential and the exact logarithmic differential of the distribution density of the Brownian motion, this density is formed by in-

[6] Kozyrev was a reknown astronomer of his time, he predicted the volcano eruptions that were observed in the Moon in the late fifties. He was imprisoned in a gulag for ten years under the charges of ”sabotage on astronomy”; two months after his release in 1947 he completed his doctoral thesis from which stemmed his work.

[7] In a logophysical approach associated to Matrix Logic, that is produced by the non-duality of the True and False *operators* -in contrast with *scalar* Boolean values- that produces a torsion arising from their non-trivial commutator that is associated with the Klein bottle, we encounter a time *operator* which is more general than the one envisioned as a mere parameter in physics. This time is the one sustaining the chronomes of which the Kozyrev phenomenae are an example between the many others that encompass number theory, the continuum hypothesis in mathematics, the Mendeleiev periodic table of elements, the fixed action patterns that are central to the human body physiology in neurosciences, etc. We refer the reader to the studies by Rapoport [75, 77].

terpolation between the initial and final distributions of the density, which by the way, form the Schroedinger wave function. In this perspective, non-relativistic quantum mechanics which is designed in terms of the time variable which coincides with the time-universal parameter, has the same features that these remarkable processes observed by Kozyrev and followers, it incorporates the past and the future into its setting. While being formally time-symmetric, the Schroedinger equation admits a realization in terms of the future evolving Brownian process built from a RCW connection in which the Schroedinger field is part of its drift through the gradient of its logarithm. As we have seen already, we can take the Hodge dual of the trace-torsion, say, the one that gives the drift of this non-relativistic Brownian motion, and thus obtain a pseudo-three-form that may be associated with the angular momentum field characteristic of the experiments carried out by Kozyrev.

4. The Hodge Decomposition of the Trace-Torsion Field

To obtain the most general form of the RCW laplacian in the non-degenerate case, we only need to know the most general decomposition of 1-forms. To start with, in this section, we have a smooth orientable n-manifold M provided with a Riemannian metric g. We consider as above, the Hilbert space given by the completion of the pre-Hilbert space of square-integrable smooth differential forms of degree k $(0 \leq k \leq n)$ on M, with respect to the Riemannian volume $\mathrm{vol}_{\mathrm{g}}$, which we denote as $L^2(\sec(\Lambda^k(T^*M))$. We shall focus on the decomposition of 1-forms, so let $\omega \in L^2(\sec(T^*M))$; then we have the Hilbert space decomposition

$$\omega = df + A_{\mathrm{coex}} + A_{\mathrm{harm}}, \tag{16}$$

where f is a smooth real valued function on M, A_{coex} is a smooth coexact 1-form, i.e. there exists a smooth 2-form, β such that $\delta\beta_2 = A_{\mathrm{coex}}$, so that A_{coex} is coclosed, i.e.

$$\delta A_{\mathrm{coex}} = \delta(\delta\beta_2) = 0, \tag{17}$$

and A_{harm} is a closed and coclosed smooth 1-form, then

$$\delta A_{\mathrm{harm}} = 0, \quad dA_{\mathrm{harm}} = 0, \tag{18}$$

or equivalently, A_{harm} is harmonic, i.e.

$$\triangle_1 A_{\mathrm{harm}} \equiv \mathrm{trace}(\nabla^g)^2 A_{\mathrm{harm}} - R^{\alpha}_{\beta}(g)(A_{\mathrm{harm}})_{\alpha}\gamma^{\alpha} = 0, \tag{19}$$

with $R^{\alpha}_{\beta}(g) = R^{\mu\beta}_{\mu\alpha}(g)$ the Ricci metric curvature tensor. Eq. (16) is the sourceless Maxwell-de Rham equation. An extremely important fact is that this is a Hilbert space decomposition, so that it has unique terms, which are furthermore orthogonal in Hilbert space, i.e.

$$((df, A_{\mathrm{coex}})) = 0, \quad ((df, A_{\mathrm{harm}})) = 0, \quad ((A_{\mathrm{coex}}, A_{\mathrm{harm}})) = 0, \tag{20}$$

so that the decomposition of 1-forms (as we said before, this is also valid for k-forms, with the difference that f is a $k-1$-form, β_2 is really a $k+1$-form and A_{harm} is a k-form) has unique terms, and a fortiori, this is also valid for the Cartan-Weyl 1-form. We have proved that $A_{\rm coex}$ and $A_{\rm harm}$ are further linked with Maxwell's equations, both for Riemannian and Lorentzian metrics. For the stationary state which we shall describe in the next section, they lead to the equivalence of the Maxwell equation and the relativistic quantum mechanics equation of Dirac-Hestenes in a Clifford bundle setting [2,49] whenever the coclosed (Hertz potential) term and the Aharonov-Bohm harmonic term are both dependent on all the 4D variables while they are infinitesimal rotations defined on the spin-plane. Furthermore, in regards to the above mentioned classical theory of the electron due to Gill, Zachary and Lindesay [25], the validity of this decomposition in a Riemannian metric, say , Euclidean space, points to the validity of having this theory of Brownian motion formulated in a non-degenerate (albeit trivial) space-time: these electromagnetic potential terms can be associated with a classical electron which does not require a quantum treatment and allows the introduction of a global time parameter. Furthermore, by studying the topological dimension of the trace-torsion, i.e. the irreducible number of minimal dimensions in space and time on which its coefficients are dependent, we can introduce helicity, spinor structures, minimal surfaces associated to them, superconductivity, turbulence and coherent structures, in short, a topological theory of processes, following the studies by R. Kiehn [33]. Thus, in this approach we can introduce spinor structures on looking to the topological features of the trace-torsion.

4.1. The Decomposition of the Cartan-Weyl Form and The Stationary State

We wish to elaborate further on the decomposition of Q in the particular state in which the diffusion process generated by $H_0(g,Q)$ and its extensions to differential forms, in the case M has a Riemannian metric g, and has a τ-invariant state corresponding to the asymptotic stationary state. Thus, we shall concentrate on the diffusion processes of scalar fields generated by

$$H_0(g,Q) = \frac{1}{2}(\triangle + L_{\hat{Q}}), \quad \text{with} \quad Q = d\ln\psi^2 + A_{\rm coex} + A_{\rm harm}. \tag{21}$$

This is the invariant form of the (forward) Fokker-Planck operator of this theory (and furthermore of the Schroedinger operator when introducing the phase function to the exact term of Q). Through this identification, we note that ψ is the scale field in the Einstein λ transformations from which in the vacuum, the RCW geometry can be obtained; see [2]. We are interested now in the $\mathrm{vol_g}$-adjoint operator defined in $L^2(\sec(\Lambda^n(T^*M)))$, which we can think as an operator on densities, ϕ. Thus,

$$H_0(g,Q)^\dagger\phi = \frac{1}{2}(\triangle_g\phi - \mathrm{div_g}(\phi\mathrm{grad}\ \ln\phi) - \mathrm{div_g}(\phi\hat{A})). \tag{22}$$

The operator described by eq. (20) is the backward Fokker-Planck operator. The transition density $p^\nabla(\tau,x,y)$ is determined by the fundamental solution (i.e. $p^\nabla(\tau,x,-)\to\delta_x(-)$ as $\tau\to 0^+$) of the equation on the first variable

$$\frac{\partial u}{\partial\tau} = H_0(g,Q)(x)u(\tau,x,-). \tag{23}$$

Then , the diffusion process $\{x(\tau) : \tau \geq 0\}$, gives rise to the Markovian semigroup $\{P_\tau = \exp(\tau H_0(g,Q)) : \tau \geq 0\}$ defined as

$$(P_\tau f)(x) = \int p^\nabla(\tau,x,y) f(y) \mathrm{vol}_g(y). \tag{24}$$

It has a unique τ-independant-invariant state described by a probability density ρ independent of τ determined as the fundamental weak solution (in the sense of the theory of generalized functions) of the τ-independent Fokker-Planck equation:

$$H_0(g,Q)^\dagger \rho \equiv \frac{1}{2}(-\delta d\rho + \delta(\rho Q)) = 0. \tag{25}$$

Let us determine the corresponding form of Q, say $Q_{\mathrm{stat}} = d\ln\psi^2 + A_{\mathrm{stat}}$. We choose a smooth real function U defined on M such that

$$H_0(g,Q_{\mathrm{stat}})^\dagger(e^{-U}) = 0, \tag{26}$$

so that

$$-de^{-U} + e^{-U}Q = \delta(-\delta\Pi + A_{\mathrm{harm}}), \tag{27}$$

for a 2-form Π and harmonic 1-form A_{harm}; thus, if we set the invariant density to be given by $\rho = e^{-U}\mathrm{vol}_g$, then

$$Q_{\mathrm{stat}} = d\ln\psi^2 + \frac{A}{\psi^2}, \quad \text{with} \quad A = -\delta\Pi_2 + A_{\mathrm{harm}}. \tag{28}$$

Now we project $\frac{A}{\psi^2}$ into the Hilbert-subspaces of coexact and harmonic 1-forms, to complete thus the decomposition of Q_{stat} obtaining thus Hertz and Aharonov-Bohm potential 1-forms for the stationary state respectively. Yet these potentials have now a built-in dependence on the invariant distribution, and although they give rise to Maxwell's theory, the interpretation is now different. [8] Indeed, we have an inhomogeneous random media, and these potentials depend on the τ-invariant distribution of the media. It was proved in [65] that these potentials appear in the context of the equivalence of the Maxwell sourceless equation on Minkowski space written in terms of a Dirac-Hestenes spinor field, and the non-linear Dirac-Hestenes equation for these fields, albeit in Minkowski space provided with a RCW connection with trace-torsion given by Q_{stat}. Yet, we can exploit further the Hodge-decomposition of Q_{stat} to manifest the quantum potential as built-in. Indeed, if we multiply it by ψ and apply ∂, then we get that $d\ln\psi$, and the coexact and harmonic terms of Q_{stat} decouple in the resultant field equation which turns out to be

$$\triangle_g\psi = [g^{-1}(d\ln\psi, d\ln\psi) - \delta d\ln\psi]\psi, \tag{29}$$

with nonlinear potential $V := g^{-1}(d\ln\psi, d\ln\psi) - \delta d\ln\psi$, which has the form of (twice) a relativistic quantum potential extending Bohm's potential in non-relativistic quantum mechanics [7].

[8] A word of caution. In principle, $-\delta\Pi/\rho$ and A_{harm}/ρ may not be the coexact and harmonic components of A/ρ respectively. If this would be the case, then we obtain that $d\ln\psi$ is g^{-1}-orthogonal to both $-\delta\Pi$ and A_{harm}; furthermore $d\ln\psi \wedge A_{\mathrm{harm}} = 0$, so furthermore they are collinear. This can only be for null A_{harm} or constant ρ, so that the normalization of the electromagnetic potentials is by a trivial constant. In the first case the invariant state has the sole function of determining the exact term of Q to be (up to a constant) $d\ln\psi$.

4.2. Time Reversal Invariance of the Brownian Motions, the Torsion Potential and Detailed Balance

Finally, we want to recall the essence of the problem of time-invariance of the diffusion processes on the invariant state. In this setting, following the well known Kolmogorov characterization of irreversibility of Brownian motions [35], τ-reversibility for a diffusion process generated by $H_0(g,Q)$ is verified whenever for any two smooth compact supported functions f,h defined on M, we have that

$$\int_M u(H_0(g,Q)v)\rho \mathrm{vol_g} = -\int_M g(\nabla u, \nabla v)\rho \mathrm{vol_g} = \int_M v(H_0(g,Q)u)\rho \mathrm{vol_g}. \tag{30}$$

This is satisfied if and only if

$$Q = \frac{1}{2} d\ln\rho = d\ln\psi, \tag{31}$$

or still, the electromagnetic terms of the torsion one-form vanish. In a general setting, these terms have been associated to rotational Coriolis forces by [68], and in the classical treatise on statistical thermodynamics by Lavenda [42].

5. Energy Forms, the Quantum Potential and RCW Diffusions

In this section we shall show that the RCW geometries yield a natural formulation of quantum mechanics on manifolds, as an operator theory on two Hilbert spaces [13]. So, this section and the next, we will discuss basic issues which on the usual setting have been somehow obviated and are far from being obvious. The basic formalism which leads to this is the well known remarkable correspondence explored in flat Euclidean space between the Dirichlet forms of potential theory, Markovian semigroups and their diffusion processes [13] and RCW laplacian operators [65], and originates in the canonical commutation relations. In fact, in quantum field theory on curved space-time, the starting point is an energy functional for the field associated to a self-adjoint operator on the Hilbert space determined by the Riemannian volume element [11]. In our theory, this self-adjoint operator will appear to be the conformal transform of the self-adjoint extension of the RCW laplacian as defined on an adequate subspace of the ground-state Hilbert space with a weighted inner product defined by the invariant density. Thus, two Hilbert spaces are needed: the ground-state Hilbert space on which we have a diffusion generated by the RCW laplacian which acts as the Fokker-Planck operator, and the Hilbert space defined by the Riemannian volume in which this operator transforms into the Schroedinger operator. We shall present below the above mentioned correspondences.

We assume that M has a Riemannian metric; we assume further that is four-dimensional space-time (and thus, we are in the situation discussed in [65] and references therein) and a diffusion process with stationary state $\psi^2 \mathrm{vol_g}$ with null electromagnetic terms in eq. (28), generated by $H_0(g, d\ln\psi^2) = \frac{1}{2}(\triangle_g + \mathrm{grad}\ln\psi^2)$, a Hamiltonian operator on the Hilbert space $L^2(\psi^2 \mathrm{vol}_g)$; thus, the drift vector field is $\mathrm{grad}\ln\psi$. With abuse of notation, let us denote still as $H_0(g, d\ln\psi^2)$ the Friedrichs self-adjoint extension [13] of the infinitesimal generator given in eq. (27) with domain given by $\mathcal{D}$, the space of compact supported infinitely differentiable functions on M. We can now define the inner product

$$(f_1, f_2)^{\rho} = 1/2 \int g^{-1}(df_1, df_2)\psi^2 \mathrm{vol}_g \tag{32}$$

By integration by parts, we obtain

$$(f_1, f_2)^{\rho} = -(f_1, H(g, d\ln\psi^2) f_2)^{\rho} \tag{33}$$

where $(.,.)^{\rho}$ denotes the weighted inner product in $L^2(\psi^2 vol_g)$. Let us consider now the closed quadratic form, (the *Dirichlet form*) q associated to $(.,.)^{\rho}$, i.e. $q(f) = (f, f)^{\rho}$. We see from eq.(32) that there is a unique Hamiltonian operator which generates q, it is the self-adjoint operator $-H_0(g, d\ln\psi^2)$. Since the quadratic form is positive, $q(f) \geq 0$, for any $f \in L^2(\psi^2 vol_g)$, then $H_0(g, d\ln\psi^2)$ is a negative self-adjoint operator on $L^2(\psi^2 vol_g)$ and the Markovian semigroup $exp(\tau H(g, d\ln\psi^2))$ is defined. Let us see how this construction is related to the usual formulation of Quantum Mechanics in terms of quadratic forms in $L^2(\mathrm{vol}_g)$, which in the non-relativistic flat case has been elaborated by several authors [13]. Consider the mapping $C_{\psi} : L^2(\psi^2 \mathrm{vol}_g) \to L^2(vol_g)$ defined by multiplication by ψ; this is the groundstate transformation and defines a conformal isometry between the two Hilbert spaces. This map takes $C_0^{\infty}(M)$ into itself. For any f in $C_0^{\infty}(M)$ we have

$$\begin{aligned} q(\psi^{-1}f) &= (\psi^{-1}f, \psi^{-1}f)^{\rho} \\ &= 1/2 \int \{g^{-1}(df, df) - 2g^{-1}(df, d\ln\psi)f + g^{-1}(d\ln\psi, d\ln\psi)f^2\}\mathrm{vol}_g \\ &= 1/2 \int \{g^{-1}(df, df) + (\mathrm{div}_g(b)f^2 + g(b, b)f^2\}\mathrm{vol}_g \\ &= \int f\Big\{-\frac{1}{2}\triangle_g + V\Big\} f \mathrm{vol}_g = (f, Hf)_{L^2(vol_g)} \end{aligned} \tag{34}$$

where we denoted $b = \mathrm{grad}\ \ln\psi$ which is the drift vector field (denoted initially as $\hat{Q}$, the g-conjugate of the trace-torsion one-form) of the process generated by $H_0(g, d\ln\psi^2)$ since by eqs. (21, 28) this is $\frac{1}{2}\mathrm{grad}\ \ln\psi^2$ and

$$H = C_{\psi} \circ H(g, d\ln\psi^2) \circ C_{\psi}^{-1} = -1/2\triangle_g + V, \tag{35}$$

where in the weak sense,

$$V = \frac{1}{2}(\mathrm{div}_g\, b + g(b, b)) = \frac{\triangle_g \psi}{2\psi}, \tag{36}$$

is the *relativistic quantum potential*; here, in distinction with Bohm's quantum potential in non-relativistic Quantum Mechanics [7] (which is retrieved in the case of $n = 3$ and g the Euclidean metric), it depends on both the space and time-t coordinates. Then, we have proved that $-H(g, d\ln\psi^2)$ is unitarily equivalent to the Hamiltonian operator $H := -\frac{1}{2}\triangle_g + V$ defined on $L^2(\mathrm{vol}_g)$ and ψ is a generalized groundstate eigenfunction of H with 0 eigenvalue. The non-linear dependence of V on the invariant density introduced by ψ introduces non-local correlations on the quantum system We shall see below that this dependence of V on ψ is removed due to conformal invariance. This will establish that the Schroedinger operator H has for quantum potential one-twelfth of the Riemannian scalar metric and thus H coincides with the Riemannian conformal invariant wave operator considered in quantum gravity in curved spaces [11]). We shall now elaborate on these aspects.

6. The Mean Curvature Extremal Principle

Since at the level of constitutive equations for Q, the electromagnetic potentials decouple from the ψ-field (see the discussion that lead to eq. (29)) we can study independently the field equations from which the RCW connection with exact Q can be derived. We shall assume that $n = 4$. We start with a general Riemann-Cartan connection (Γ^{ab}_{α}), (where Greek letters denote space-time indices as until now, and Latin letters denote anholonomic indices), and we introduce its scalar curvature

$$R(\Gamma) = e^{\alpha}_{a} e^{\beta}_{b} R^{\cdot\cdot ab}_{\alpha\beta}, \tag{37}$$

where the e^{α}_{a} is a field of invertible tetrads with $g_{\alpha\beta} = \delta_{ab} e^{a}_{\alpha} e^{b}_{\beta}$, with δ_{ab} the Euclidean metric [9], and $R^{\cdot\cdot ab}_{\alpha\beta}$ is the curvature tensor of (Γ^{ab}_{α}) [14, 65]. Now we recall the Einstein's λ transformations of above (here ρ will be substituted by a scalar field ϕ): Let ϕ be a real function on M. Then $\lambda(\Gamma^{a}_{\alpha b}) := \Gamma^{a}_{\alpha b}$, and $\lambda(e^{\alpha}_{a}) := \phi^{-1} e^{\alpha}_{a}$ so that $\lambda(g_{\alpha\beta}) = \phi^{2} g^{\alpha\beta}$ and then the scalar curvature transforms as $\lambda(R(\Gamma)) = \phi^{-2} R(\Gamma)$, and finally $\mathrm{vol}_{\lambda(g)} = \phi^{4} \mathrm{vol}_{g}$. Since the scalar fields ψ transform as $\lambda(\psi) = \phi^{-1}\psi$, we get that the functional

$$A(\Gamma, \psi, g) = \int R(\Gamma) \psi^{2} \mathrm{vol}_{g}, \tag{38}$$

is invariant by the set of λ transformations, i.e.: $A(\lambda(\Gamma), \lambda(\psi), \lambda(g)) = A(\Gamma, \psi, g)$. Notice that if from the field equations we obtain that $\psi^{2}\mathrm{vol}_{g}$ can be identified with the unique invariant density of the diffusion process generated by $H_{0}(g, d\ln\psi^{2})$, then (37) is the *mean Riemann-Cartan scalar curvature*. Taking variations with respect to g we obtain that

$$R_{\alpha\beta}(\Gamma) - 1/2 g_{\alpha\beta} R(\Gamma) = 0, \tag{39}$$

i.e. the Einstein-Cartan equations for Γ in the vacuum, while by taking variations with respect to $\Gamma^{\gamma}_{\alpha\beta}$, we obtain that torsion tensor is a particular case of the one we derive from the anticommutator of eq. (9), since we have

$$T^{\gamma}_{\alpha\beta} = \delta^{\gamma}_{\alpha}\, \partial_{\beta} \ln\psi - \delta^{\gamma}_{\beta} \partial_{\gamma} \ln\psi, \tag{40}$$

so that, up to factor of 3 which we shall absorb so we shall take $Q = d\ln\psi$ and thus the field equations have yielded a RCW structure with exact Q. Taking variations with respect to ψ

[9] All the following definitions of the λ transformations and the ensuing field equations are valid as well if we take here the Minkowski metric; since we do not know whether our construction of a relativistic Brownian motion carries from the Minkowski space to general Lorentzian metrics , in this section we shall keep the metric to be positive-definite for which we take the initial metric to be Euclidean. Brownian motions the Schwarzild metric has been recently constructed on the unit tangent manifold (see J. Franchi and Y. Le Jan,Relativistic Diffusions, arXiv:math.PR/0403499). The relation of this construction, with the Lorentz-invariant Brownian motions on Minkowski space presented in [63] and the present article is unknown.

we get the teleparallelism: $R(\Gamma) = 0$; replacing eq. (39) in eq. (38) we get the field for the Einstein metric tensor $G_{\alpha\beta}(g) = R_{\alpha\beta}(g) - \frac{1}{2}R(g)$:

$$G_{\alpha\beta}(g) = -\frac{6}{\psi^2}\partial_\alpha\psi\,\partial_\beta\psi - 1/2g_{\alpha\beta}\partial_\gamma\psi\partial^\gamma\psi - \frac{1}{6}(\nabla_\alpha\nabla_\beta\psi^2 - g_{\alpha\beta}\triangle_g\psi^2), \tag{41}$$

where in the r.h.s. we identify (up to a factor) minus the improved energy-momentum density of the scalar field in renormalizable gauge theories. Now, by taking the trace in this equation we finally get

$$\Big(\triangle_g - \frac{1}{6}R(g)\Big)\psi = 0, \tag{42}$$

so that ψ is a generalized groundstate of the conformal invariant wave operator defined on $L^2(\mathrm{vol}_g)$. Note that from eqs. $(34,35,41)$ we conclude that the quantum potential is $\frac{1}{12}R(g)$ which *does not* depend on the scalar field ψ at all. Therefore, the correlations on the quantum system under Brownian motion with drift given by $b = \mathrm{grad}\ \ln\psi$ are mediated by the metric scalar curvature (which, of course, does not depend on ψ any more; this is the *form invariance* of the quantum potential [7])! Otherwise stated and in view of the relation between the noise tensor and the Riemannian metric (see the discussion after eq. (15)), when we have an anisotropic noise tensor we have constructed a non-trivial metric and quantum non-local correlations which are due to the metric scalar curvature.

Solving the conformal invariant wave equation with Dirichlet regularity conditions on the closure of an open neighborhood of M [13], we obtain a conformally conjugate Dirichlet form whose associated Hamiltonian operator is $-H_0(g, d\ln\psi^2)$, with ψ a solution of eq. (41) and thus the Markovian semigroup determined by it can be reconstructed by reversing the steps in the previous Section. We shall finally establish the relation between the heat kernel $p_{\mathrm{conf}}(\tau,x,y)$ of the Markovian semigroup $exp(\frac{\tau}{2}H)$ and the heat kernel $p_\psi(\tau,x,y)$ of the RCW semigroup. We have

$$\begin{aligned} exp(\tau H_0(g,d\ln\psi^2))f(x) &= \psi^{-1}(x)exp(\frac{\tau}{2}H)(\psi f)(x) \\ &= \int \psi^{-1}(x)p_{\mathrm{conf}}(\tau,x,y)\phi(y)f(y)vol_g(y) \end{aligned} \tag{43}$$

so that we conclude that

$$p_\psi(\tau,x,y) = \psi^{-1}(x)\psi(y)p_{\mathrm{conf}}(\tau,x,y). \tag{44}$$

Thus, we have linked the kernels of the quantization in the two Hilbert spaces, the groundstate Hilbert space $L^2(\psi^2 vol_g)$, and $L^2(vol_g)$. The former corresponds to the RCW geometry, while the latter is the usual Hilbert space for the quantization of the kinetic energy of a spinless massive free-falling test-particle, in terms of the Riemannian invariants of the manifold M described in terms of g! We remark that the introduction of both spaces and the unitary transformation between them, has allowed us to identify the quantum potential, while working only in the usual Hilbert space would not have allowed for this identification; finally, the scalar curvature term so much discussed has been found to be a resultant of the λ invariance of the theory, and not the resultant of technicalities in computing the propagators; as discussed already in [65], this theory has no ordering problem [62]. Thus, in the $L^2(vol_g)$ space we have found the Hamiltonian operator considered by B.de Witt, and reencountered by several researchers in quantum field theory in Riemannian geometries through the short-τ expansion of $p_{\rm conf}(\tau,x,x)$ [11] in geometrical and topological invariants , and for the path integral representations for Fokker-Planck operators [41], which as we already saw, when g is Riemannian, are precisely of the form $H_0(g,Q)$. Yet, our result is in disagreement with the path integral representation of the classical kinetic energy of a massive particle in a Riemann-Cartan geometry due to Kleinert, in which he obtains *twice* the quantum potential (see, chap. X, [34]).

7. RCW Diffusions and Nonrelativistic Quantum Mechanics

From the previous section we know that for the stationary state defined by ρ the laplacian defined by a RCW connection is symmetric with respect to the measure defined by ρ if and only if the trace-torsion is given by $Q = \frac{1}{2}d\ln\rho$. Futhermore, it is a non- positive-definite operator since for any functions in the space $\mathcal{D}$ of compact supported functions u and v defined on M we have the Green identity

$$\int_M u(H_0(g,Q)v)\rho\mathrm{vol_g} = -\int_M g(\nabla u,\nabla v)\rho\mathrm{vol_g} = \int_M v(H_0(g,Q)u)\rho\mathrm{vol_g}. \tag{45}$$

We wish to see if there exists a self-adjoint extension of $H_0(g,Q)|_{\mathcal{D}}$ in the space $L^2 = L^2(M,\rho)$ of square-integrable functions with respect to the density $\rho\mathrm{vol_g}$. Consider the space $W^1(M,\rho) = \{f : M \to C, f \in L^2, \nabla f \in L^2\}$ where we mean by ∇f the distributional gradient. We can turn this space into a complex Hilbert space by working with complex-valued functions provided with the inner product

$$(u,v)_{W^1} = \int_M u\bar{v}\rho vol_g + \int_M g(\nabla u,\nabla\bar{v})\rho\mathrm{vol_g}. \tag{46}$$

Let W_0^1 be the closure of $\mathcal{D}$ in W^1; define $W_0^2 = W_0^2(M,\rho) = \{f \in W_0^1/ H_0(g,d\ln\rho)f \in L^2\}$ where the latter action of the operator is meant in the distributional sense. Since $\mathcal{D} \subset W_0^2$, then $H_0(g,d\ln\rho)|_{W_0^1}$ is an extension of $H_0(g,d\ln\rho)|_{\mathcal{D}}$. Therefore $-H_0(g,d\ln\rho)$ is a positive-definite self-adjoint extension defined in L^2. Furthermore, if M is geodesically complete, then $H_0(g,d\ln\rho)|_{W_0^2}$ is a unique self-adjoint extension of $H(g,d\ln\rho)|_{\mathcal{D}}$. [10]

Consider next the Dirichlet problem for $H_0(g,Q|_{W_0^2}$ on a relatively compact non-empty set Ω in M, so that

$$\begin{cases} H_0(g,Q)u+\lambda u &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \partial\Omega, \end{cases}$$

where λ is constant. This can be considered in the weak sense: We look for a non-zero function $u \in W_0^1(\Omega,\rho)$ such that for all $v \in W_0^1(\Omega,\rho)$,

$$-\int_\Omega g(\nabla u,\nabla v)\rho\mathrm{vol_g} + \lambda\int_\Omega uv\rho\mathrm{vol_g} = 0. \tag{47}$$

[10] A short approach to the proof. The first part is the fact that the space W_0^1 is a Hilbert space so that the quadratic form $\mathcal{E}(u,v) = \int_M g(\nabla u,\nabla v)\rho\mathrm{vol_g}$ with the domain W_0^1 is closed in L^2. Therefore, it has the generator, which is self-adjoint with domain W_0^2 and hence, is the Friedrich extension of $H(g,\frac{1}{2}d\ln\rho)|_{\mathcal{D}}$; see [20].

It is easy to prove that u is a solution of this problem if and only if $u \in W_0^2(\Omega,\rho)$ and $H_0(g,Q)u+\lambda u=0$. Considering then the manifold Ω provided with the density ρ, we conclude that the eigenvalues of the weak Dirichlet problem in Ω are exactly the eigenvalues of the self-adjoint operator $-H(g,Q)|_{W_0^2(\Omega,\rho)}$ in $L^2(\Omega,\rho)$.

We have a theorem due to Rosenberg [37]: For any non-empty relatively compact open set $\Omega \subset M$, the spectrum of $-H(g,Q)|_{W_0^2(\Omega,\rho)}$ is discrete and consists of a sequence $\{\lambda_k(\Omega)\}_{k=1}^{\infty}$ of non-negative real numbers such that $\lambda_k(\Omega)\to\infty$ as $k\to\infty$. If in addition $M-\bar{\Omega}$ is non-empty, then $\lambda_1(\Omega)>0$.

Assuming that the eigenvalues are counted with multiplicity, we have the Weyl asymptotic formula

$$\lambda_k(\Omega) \approx c_n\left(\frac{k}{\int_\Omega \rho\mathrm{vol}_g}\right)^{\frac{2}{n}}, \quad \text{as} \quad k\to\infty, \tag{48}$$

where $n=\dim(M)$ and the constant $c_n>0$ is the same as in R^n.

If M is compact, then we have $\lambda_1(M)=0$, because the function $f=$ constant is an eigenfunction. Since $H_0(g,Q)f=0$ implies $f=0$ (we are assuming that M is connected), the multiplicity of the bottom eigenvalue is 1 and then $\lambda_2(M)$ is strictly positive. In any case, the lowest eigenvalue of $-H_0(g,Q)|_{W_0^2(M,\rho)}$ can be determined as follows.

Furthermore, we have a theorem (Rayleigh Principle) for the minimal eigenvalue [20]: For a manifold M provided with a density $\rho\mathrm{vol_g}$,

$$\lambda_{min}(M)=\inf_{f\in\mathcal{T}-0}\frac{\int_M|\nabla f|^2\rho\mathrm{vol_g}}{\int_M f^2\rho\mathrm{vol_g}}, \tag{49}$$

where $\mathcal{T}$ is any class of test functions such that $\mathcal{D}\subset\mathcal{T}\subset W_0^1$.

Proof: It follows from the variational principle for the operator $-H_0(g,Q)|_{W_0^2}$ and by the Green formula, that

$$\begin{aligned}\lambda_{\min}(M) &= \inf_{f\in W_0^2}\frac{-\int_M(H_0(g,Q)f,f)\mathrm{vol_g}}{||f||_{L^2}^2} && (50)\\ &= \inf_{f\in W_0^2-0}\frac{\int_M|\nabla f|^2\rho\mathrm{vol_g}}{||f||_{L^2}^2}, && (51)\end{aligned}$$

and by observing that $\mathcal{D}\subset W_0^2\subset W_0^1$ and $\mathcal{D}$ is dense in W_0^1.

8. Geometric-Stochastic Quantum Mechanics on State-Space

We consider a complex separable Hilbert space $\mathcal{H}$ and a self-adjoint operator H defined on $\mathcal{H}$. The time development of quantum systems is given by the one-parameter group $\{e^{-itH},t\in R\}$ of unitary operators. A pure quantum state $\psi\in\mathcal{H},||\psi||=1$, develops according to

$$\psi_t=e^{-itH}\psi \tag{52}$$

which can be reformulated in terms of the Schroedinger equation

$$\frac{\partial\psi_t}{\partial t}=-iH\psi_t. \tag{53}$$

Still, pure quantum states are described by equivalence classes $[\psi]$ of unit vectors $\psi \in \mathcal{H}$, where two vectors are equivalent if they differ by a complex phase factor. Then, the time development of the state $[\psi]$ is given by

$$\Phi_t([\psi]) = [e^{-iHt}\psi]. \tag{54}$$

While eqs. (51) and (52) are equivalent, this is no longer the case of eqs. (52) (53), since ψ_t can contain a complex time dependant factor. The proper setting for quantum mechanical evolution in terms of the Schroedinger equation requires to take in account this indeterminate factor. So the state space is the projective Hilbert space $\mathcal{P}(\mathcal{H})$, and the time evolution of quantum systems are curves on this space of the form $\gamma(t) = \Phi_t([\psi]) = [e^{-iHt}\psi]$.

There are two ways in which one can construct from a heat semigroup defined by a RCW diffusion process its quantum Schroedinger representation. In this case, the hamiltonian operator is $H(g,Q)$ associated to a RCW connection with $Q = \frac{1}{2}d\ln\rho$ and the corresponding unitary group $[e^{-i\tau H(g,Q)}]$ defined on the natural complex extension of a real Hilbert space as we have taken in the previous section, corresponds to the so-called Euclidean analytical Schroedinger representation for the diffusion semigroup defined by this space-time structure, yet with some differences we would like to remark. Firstly, there is a freedom upon the choice of the time, it can be τ for a relativistic theory in which g can depend on t as well as Q and our space-time manifold is a 4-dimensional manifold , M, or, we can write down a non-relativistic theory, for which τ and t coincide [71] and space-time is $R \times M$ where M is a 3-manifold, but still we have in this foliated manifold a Riemannian metric which may depend on t as well as the trace-torsion Q; in any case, due to the fact that in Quantum Mechanics observables are self-adjoint operator (real eigenvalues) we have to restrict Q to be exact of the form $Q = \frac{1}{2}d\ln\rho$ because the inclusion of the electromagnetic terms, following the Kolmogorov characterization of τ-symmetric diffusion processes, produces $H(g,Q)$ for general Q to be a non-symmetric operator in $L^2(M,\rho)$, so we cannot introduce the self-adjoint extension of it. The other possibility is to develop a covariant formulation of non-relativistic Quantum Mechanics in $R \times M$ in which we transform the diffusion processes into the Schroedinger equation without applying the Euclidean time scheme, but in this case Q does not necessarily restrict to the exact differential term, including thus the electromagnetic terms and the Schroedinger operator is associated to the RCW laplacian in an indirect way in which will adquire the form $\triangle_g + V$, where V is a potential which can eventually depend on the wave function or not,which for appropiate classes of potentials V can result in a self-adjoint operator; see page 34 in Schechter [87]. Both theories we know already how to formulate as an infinite-dimensional Hamiltonian system (in the sense of classical mechanics), as long as the spectrum of $H(g,Q)$ is discrete, which in the case of Q restricted to be exact, is already the case as discussed above. In this article, we shall present both alternatives. Finally, having set the geometric approach to quantum mechanics in Hilbert space, we can further study the so-called stochastic extension of the Schroedinger equation, which amounts to write the s.d.e. which extend the Hamiltonian flow with a noise term which drives the system to a particular eingenstate, providing thus for the reduction of the wave function.

9. The State-Space RCW Quantum Geometries, Brownian Motions and the Reduction of the Wave Function

The notion of a geometric theory of quantum mechanics has been in most of the works associated with the idea of placing in a purely geometrical context the operator formalism of quantum mechanics and describing the processes of observation in terms of geometrical distance in state-space; the other approach that can be named identically as quantum geometry, is the present approach that is valid for both configuration manifolds and state-space manifolds . The former geometrical approach has lead to formulate non-relativistic quantum mechanics as a theory of Kahler manifolds, and to breach the gap with classical mechanics which as well known, is formulated in terms of symplectic flows, and in particular, those associated with a Hamiltonian function independent of time. The Hamiltonian function that generates the Schroedinger flow is non other that the expectation value function defined on state-space of the quantum Hamiltonian operator. In this so called quantum geometry (see [29, 86] and references therein), the Schroedinger equation is a symplectic flow in state-space, given by a complex projective manifold, provided with the Fubini-Study metric, with its naturally associated symplectic and Kahlerian structures. Furthermore, by considering random perturbations of this symplectic flow to account for the role of the environment in the quantum system, the reduction of the wave-function has been described in terms of stochastic processes on the quantum geometry on state-space [29]. This approach to the so-called *open* Schroedinger equation has been elaborated as an emergent theory of a background statistical theory of unitary matrices. In none of this approaches to the open Schroedinger equation, no relation was established with the fact that there is a quantum geometry in space-time and its association with Brownian motions. Thus, this chapter aims to present a very short account of the fact that we can describe the stochastic processes in state-space that describe the reduction of the wave-function in terms of the same stochastic-geometrical structures of Riemann-Cartan, and that the Schroedinger symplectic flow defined by the expectation value of the Hamiltonian operator is (up to a modification which drives the measurement process to a specific eigenstate) the natural choice for the drift. In particular,one can start with a stochastic differential equation, consider the connection on space-time defined by it and its differential generator which is the Laplacian operator of this geometry, and study the reduction of the wave function of the quantum evolution of this space-time operator. In this sense, the role of space-time structures in producing the reduction of the wave function. So in this case, we have a two layer structure of RCW type, one related to the diffusion process in space-time and the second one, with the diffusion process in state-space that follows when studying the spectra of the RCW laplacian, or can be carried out independently for an arbitrary quantum system described by its Hamiltonian operator. In the following we shall present both quantum geometries in a single setting. In the following we follow the discussion in [70,71].

Let us assume we have a Hilbert space with finite dimension $n+1$ so we are dealing with M being $CP(n)$, the complex projective space of dimension n, the space of rays of Quantum Mechanics, although the more general infinite-dimensional case is also possible. In fact, this space not only carries a Riemannian metric, the Fubini-Study (FS) metric, which we denote as g but also a symplectic two-form Ω and still an almost complex structure provided by an endomorphism $J_z : T_zM \to T_zM$ such that $J^2 = -I$ and

$g(u,Jv)=\Omega(u,v)$ for all $u,v\in CP(n), z\in M$. Indeed, denote the hermitean product of the the $n+1$-dimensional Hilbert space of the quantum system as $<u,v>=g(u,v)+i\Omega(u,v)$ where $g(u,v)=\Re<u,v>$ and $\Omega(u,v)=\Im<u,v>$, and $g(u,v)=g(Ju,Jv)$. Furthermore J is compatible with g, i.e. $\nabla J=0$, where ∇ is the Levi-Civita covariant derivative. Thus, M provided with (g,Ω,J) becomes a Kahler manifold. For a self-adjoint Hamiltonian operator H defined on $CP(n)$, we define the quantum-expectation value function $(H):CP(n)\rightarrow R$ by $(H)(u)=\frac{<u,Hu>}{<u,u>}$. In this section we shall then restrict ourselves to the Euclidean technique and take $H=H(g,Q)$ the self-adjoint operator defined in a finite-dimensional complex subspace of the Hilbert space $\mathcal{H}=W_0^2$; this amounts to fixing a cut-off which one can fix in accordance with the estimates given above [11]. We denote the general state vector by $|z>$ with z standing for the complex projections $z^0,\ldots,z^n$ of $|z>$ on an arbitrary fixed basis. Thus, $(H)(|z>)=\frac{<z|H|z>}{<z,z>}=\frac{\bar{z}^\alpha H_{\alpha\beta}z^\beta}{\bar{z}^\delta z^\delta}$. Since (H) is homogeneous of degree zero on both $z^\alpha,\bar{z}^\alpha$ we define the new complex coordinates $t^j=\frac{z^j}{z^0}$ and $\bar{t}^j=\frac{\bar{z}^j}{\bar{z}^0}$, $j=1,\ldots,n$, which are well defined whenever $z^0\neq 0$. The real manifold structure of $CP(n)$ is defined by taking the coordinate system $(x^a), a=1,\ldots,2n$ with $x^1=\Re t^1, x^2=\Im t^2,\ldots,x^{2n-1}=\Re t^{2n-1}, x^{2n}=\Im t^{2n}$. Thus, the specification of the $2n$-vector (x^a) determines the unique ray containing the unnormalized state $|z>$. The FS metric $g=(g_{\alpha\beta}dz^\alpha\otimes d\bar{z}^\beta)$ with $g_{\alpha\beta}=4\frac{\partial^2}{\partial z^\alpha\partial\bar{z}^\beta}\ln\bar{z}^\gamma z^\gamma$ written on the real manifold is $g=(g_{ab}dx^a\otimes dx^b)$ with (see [69].)

$$g_{ab}=4\frac{[(1+x^dx^d)\delta_{ab}-(x^ax^b+\omega_{ac}x^c\omega_{bd}x^d)]}{(1+x^lx^l)^2}, \tag{55}$$

with inverse

$$g^{ab}=\frac{1}{4}(1+x^lx^l)(\delta_{ab}+x^ax^b+\omega_{ac}x^c\omega_{bd}x^d). \tag{56}$$

where ω_{ab} is a skewsymmetric tensor whose only non-vanishing terms are $\omega_{a=2j-1b=2j}=1, \omega_{a=2j,b=2j-1}=-1$. Furthermore, the complex structure $J=(J_a^b)$ satisfies $J_a^bJ_b^c=-\delta_a^c$, and the identities $J_a^cJ_c^dg_{cd}=g_{ab}$ and the symplectic form $\Omega=\Omega_{ab}dx^a\wedge dx^b$ satisfies $\Omega_{ab}=g_{bc}J_a^c$ with inverse $\Omega^{ab}=g^{ac}J_c^b$. Then (SE) takes the form (we take $\hbar=1$) of the Hamiltonian flow on M given by $\frac{dx}{dt}=2\Omega(\nabla(H))$, where $\nabla^af=g^{ab}\partial_bf(a=1,\ldots,2n)$ is the FS gradient of $f:M\rightarrow R$. To consider the dynamics of the quantum system under the influence of a measurement, we have to include the random variations due to the measurement. Thus, we extend the hamiltonian flow defined by the function (H), by considering the Ito s.d.e. (originally in [29])

$$dx^a=(2\Omega^{ab}\partial_b(H)+\rho^a)dt+\sigma g^{ab}\partial_b(H)dW(t) \tag{57}$$

with $\rho^a=-\frac{1}{4}\sigma^2g^{ab}\partial_bV$, where $V=g^{ab}\partial_a(H)\partial_b(H)=(H^2)-(H)^2$ is the variance of the Hamiltonian, or still, the squared energy quantum uncertainty. Thus, we have modified the drift with a term which depends on $\nabla(H)$, and still there is a noise tensor which is in this case a vector of the form $\sigma\nabla(H)$, with σ a constant, and we have a one-dimensional Wiener

[11] Actually, we can take for this operator, the self-adjoint extension of $H(g,Q)$ for $Q=d\ln\rho^{\frac{1}{2}}$ now acting on $CP(\infty)$ [8]; for the purpose of keeping this article to some length, we prefer not to deal with this more general case which does not imply major differences with the finite-dimensional case.

process ($i = 1$) in eq. (15). Thus, if we start from the s.d.e. (56), the metric that arises from the noise vector turns to be not the original FS metric g, but the contravariant tensor with components $\nabla^a(H)\nabla^b(H) = g^{ad}\partial_d(H)g^{be}\partial_e(H)$, times the factor σ^2, which on setting it to be equal to zero, we get the original SE written in $CP(n)$. Furthermore, the trace-torsion one-form $Q = Q_e dx^e$ has for components the functions $g_{ae}((2\Omega^{ab}\partial_b(H) + \rho^a) = 2J_e^b\partial_b(H) - \frac{1}{4}\sigma^2\partial_e V$, so that

$$Q = Jd(H) - \frac{\sigma^2}{4}d(V), \tag{58}$$

an exact differential up to an infinitesimal rotation. Next we consider two real-valued stochastic processes defined on terms of the solution curves $x(t) \in CP(n)$ of eq. (56), the Hamiltonian process defined by $(H)(x(t))$ and the variance process $V(x(t))$. Then, from applying the Ito formula and formulae of Kahlerian geometry, we find that $(H)(x(t))$ satisfies a s.d.e. with zero drift, more specifically, it is a square-integrable martingale on R, while the variance process is a supermartingale, the latter describing the reduction of the wave function to a particular eigenstate; see [1, 29]. In the present geometro-stochastic setting, we have associated to the reduction of the wave function in terms of the open Schroedinger equation, a geometry which is not riemannian, it has torsion given by the difference between the infinitesimal rotation of the differential of (H) and the differential of $\frac{\sigma^2}{4}V$; the metric is not the original FS, and as a covariant tensor it has a singularity whenever (H) is constant, i.e. on a fixed eigenstate, for which the flow of eq.(56), becomes constantly equal to it if choosen for initial value. For a completely different topological approach to superposition states and the collapse of the wave function, in terms of Matrix Logic and the Klein bottle, constructed in terms of the torsion in cognitive space introduced by the non-duality of True and False *operators* in this logic, we refer to Rapoport [75, 77].

10. RCW Geometries and Brownian Motions and the Schroedinger Equation

We have seen that one can represent the space and time quantum geometries for the relativistic diffusion associated with the invariant distribution, so that $Q = \frac{1}{2}d\ln\rho$, and $H_0(g,Q)$ has a self-adjoint extension for which we can construct the quantum geometry on state-space and still the stochastic extension of the Schroedinger equation defined by this operator on taking the analytical continuation on the time variable for the evolution parameter. In this section which follows the solution of the Schroedinger problem of interpolation by Nagasawa [51] interpreted in terms of the RCW geometries and the Hodge decomposition of the trace-torsion, we shall present the equivalence between RCW geometries, their Brownian motions and the Schroedinger equation. The fact that nonrelativistic quantum mechanics can be linked to torsion fields, has remained unseen till today, and we have proved this already for the stochastic Schroedinger equation. Thus, we shall now present the construction of non-relativistic quantum mechanics for the case that includes the full Hodge decomposition of the trace-torsion, so that $Q = Q(t,x) = d\ln f_t(x) + A(t,x)$ where $f(t,x) = f_t(x)$ is a function defined on the configuration manifold given by $[a,b] \times M$ (where M is provided with a metric, g), to be determined below, and $A(t,x)$ is the sum of the harmonic and co-closed terms of the Hodge decomposition of Q, which we shall write as $A(t,x) = A_t(x)$ as a time-dependent form on M. The scheme to determine f will be to manifest the time-reversal invariance of the Schroedinger representation in terms of a forward in time diffusion process and its time-reversed representation for the original equations for creation and destruction diffusion processes produced by the electromagnetic potential term of the trace-torsion of a RCW connection whose explicit form we shall determine in the sequel. From now onwards, the exterior differential, and the divergence operator will act on the M manifold variables only, for which we shall write then as $df_t(x)$ to signal that the exterior differential acts only on the x variables of M. We should remark that in this context, the time-variable t of non-relativistic theory and the evolution parameter τ, are identical [65, 67]. Let

$$L = \frac{\partial}{\partial t} + \frac{1}{2}A(t,x).\nabla = \frac{\partial}{\partial t} + H(g, A_t) \tag{59}$$

(here, for unburdening the notation we omit the subscript 0 on H that recalls that operates on scalar fields) with

$$\delta \hat{A}_t = -\mathrm{div}_g A_t = 0. \tag{60}$$

In this setting, we start with a background trace-torsion restricted to an electromagnetic potential. We think of this electromagnetic potential and the associated Brownian motion having its metric conjugate as its drift, as the background geometry of the vacuum, which we shall subsequently relate to a creation and destruction of particles and the equation of creation and destruction is given by the following equation.

Let $p(s,x;t,y)$ be the weak fundamental solution of

$$L\phi + c\phi = 0. \tag{61}$$

The interpretation of this equation as one of creation (whenever $c > 0$) and destruction ($c < 0$) of particles is warranted by the Feynman-Kac representation for the solution of this equation. Then $\phi = \phi(t,x)$ satisfies the equation

$$\phi(s,x) = \int_M p(s,x;t,y)\phi(t,y)dy, \tag{62}$$

where for the sake of simplicity, we shall write in the sequel $dy = \mathrm{vol}_g(y) = \sqrt{\det(g)}dy^1 \wedge \ldots \wedge dy^3$. Note that we can start for data with a given function $\phi(a,x)$, and with the knowledge of $p(s,x;a,y)$ we define $\phi(t,x) = \int_M p(t,x;a,y)dy$. Next we define

$$q(s,x;t,y) = \frac{1}{\phi(s,x)} p(s,x;t,y)\phi(t,y), \tag{63}$$

which is a transition probability density, i.e.

$$\int_M q(s,x;t,y)dy = 1, \tag{64}$$

while

$$\int_M p(s,x;t,y)dy \neq 1. \tag{65}$$

Having chosen the function $\phi(t,x)$ in terms of which we have defined the probability density $q(s,x;t,y)$ we shall further assume that we can choose a second bounded non-negative measurable function $\hat{\phi}(a,x)$ on M such that

$$\int_M \phi(a,x)\hat{\phi}(a,x)dx = 1, \tag{66}$$

We further extend it to $[a,b]\times M$ by defining

$$\hat{\phi}(t,y) = \int \hat{\phi}(a,x)p(a,x;t,y)dx, \quad \forall (t,y) \in [a,b]\times M, \tag{67}$$

where $p(s,x;t,y)$ is the fundamental solution of eq. (60).

Let $\{X_t \in M, Q\}$ be the time-inhomogeneous diffusion process in M with the transition probability density $q(s,x;t,y)$ and a prescribed initial distribution density

$$\mu(a,x) = \hat{\phi}(t=a,x)\phi(t=a,x) \equiv \hat{\phi}_a(x)\phi_a(x). \tag{68}$$

The finite-dimensional distribution of the process $\{X_t \in M, t \in [a,b]\}$ with probability measure on the space of paths which we denote as Q; for $a = t_0 < t_1 < \ldots < t_n = b$, it is given by

$$\begin{aligned} E_Q[f(X_a, X_{t_1}, \ldots, X_{t_n-1}, X_b)] &= \int_M dx_0 \mu(a,x_0) q(a,x_0;t_1,x_1)dx_1 q(t_1,x_1;t_2,x_2)dx_2 \ldots \\ &\quad \ldots q(t_{n-1},x_{n-1},b,x_n)dx_n f(x_0,x_1,\ldots,x_{n-1},x_n) \\ &:= [\mu_a q >> \end{aligned} \tag{69}$$

which is the Kolmogorov forward in time (and thus time-irreversible) representation for the diffusion process with initial distribution $\mu_a(x_0) = \mu(a,x_0)$, which using eq. (62) can still be rewritten as

$$\begin{aligned} &\int_M dx_0 \mu_a(x_0) \frac{1}{\phi_a(x_0)} p(a,x_0;t_1,x_1)\phi_{t_1}(x_1)dx_1 \frac{1}{\phi_{t_1}(x_1)} dx_1 p(t_1,x_1;t_2,x_2)\phi_{t_2}(x_2)dx_2 \ldots \\ &\qquad \ldots \frac{1}{\phi(t_{n-1},x_{n-1})} p(t_{n-1},x_{n-1};b,x_n)\phi_b(x_n)dx_n f(x_0,\ldots,x_n) \end{aligned} \tag{70}$$

which in account of $\mu_a(x_0) = \hat{\phi}_a(x_0)\phi_a(x_0)$ and $eq.(62)$ can be written in the time-reversible form

$$\begin{aligned} &\int_M \phi_a(x_0)dx_0 p(a,x_0;t_1,x_1)dx_1 p(t_1,x_1;t_2,x_2)dx_2 \\ &\qquad \ldots p(t_{n-1},x_{n-1};b,x_n)\phi_b(x_n)dx_n f(x_0,\ldots,x_n) \end{aligned} \tag{71}$$

which we write as

$$= [\hat{\phi}_a p >><< p\phi_b]. \tag{72}$$

This is the *formally* time-symmetric Schroedinger representation with the transition (but not probability) density p. Here, the formal time symmetry is seen in the fact that this equation can be read in any direction, preserving the physical sense of transition. This representation, in distinction with the Kolmogorov representation, does *not* have the Markov property.

We define the adjoint transition probability density $\hat{q}(s,x;t,y)$ with the $\hat{\phi}$-transformation

$$\hat{q}(s,x;t,y) = \hat{\phi}(s,x)p(s,x;t,y)\frac{1}{\hat{\phi}(t,y)} \tag{73}$$

which satisfies the Chapmann-Kolmogorov equation and the time-reversed normalization

$$\int_M dx\hat{q}(s,x;t,y) = 1. \tag{74}$$

We get

$$\begin{aligned} E_{\hat{Q}}[f(X_a,X_{t_1},\ldots,X_b)] &= \int_M f(x_0,\ldots,x_n)\hat{q}(a,x_0;t_1,x_1)dx_1\hat{q}(t_1,x_1;t_2,x_2)dx_2\ldots \\ &\quad \ldots\hat{q}(t_{n-1},x_{n-1};b,x_n)\hat{\phi}(b,x_n)\phi(b,x_n)dx_n, \end{aligned} \tag{75}$$

which has a form non-invariant in time, i.e. llegible from right to left, as

$$<< \hat{q}\hat{\phi}_b\phi_b]] =<< \hat{q}\hat{\mu}_b]], \tag{76}$$

which is the time-reversed representation for the final distribution $\mu_b(x) = \hat{\phi}_b(x)\phi_b(x)$. Now, starting from this last expression and rewriting in a similar form that in the forward process but now with $\hat{\phi}$ instead of ϕ, we get

$$\begin{aligned} &\int_M dx_0\hat{\phi}_a(x_0)p(a,x_0;t_1,x_1)\frac{1}{\hat{\phi}_{t_1}(x_1))}dx_1\hat{\phi}(t_1,x_1)p(t_1,x_1;t_2,x_2)\frac{1}{\hat{\phi}_{t_2}(x_2)}dx_2 \\ &\ldots dx_{n-1}\hat{\phi}(t_{n-1},x_{n-1})p(t_{n-1},x_{n-1};b,x_n)\frac{1}{\hat{\phi}(b,x_n)}\hat{\phi}_b(x_n)\phi(b,x_n)dx_nf(x_0,\ldots,x_n) \end{aligned} \tag{77}$$

which coincides with the time-reversible Schroedinger representation $[\hat{\phi}_a p >><< p\phi_b]$.

We therefore have three equivalent representations for the diffusion process, one the forward in time Kolmogorov representation, the backward Kolmogorov representation, both of them are naturally irreversible in time, and the time-reversible Schroedinger representation, so that we can write succintly,

$$[\mu_a q >>= [\hat{\phi}_a p >><< p\phi_b]] =<< \hat{q}\mu_b]], \text{with} \mu_a = \phi_a\hat{\phi}_a, \mu_b = \phi_b\hat{\phi}_b. \tag{78}$$

In addition of this formal identity,we have to establish the relations between the equations that have lead to them. We first note, that in the Schroedinger representation, which is formally time-reversible, we have an interpolation of states between the initial data $\hat{\phi}_a(x)$ and the final data, $\phi_b(x)$. The information for this interpolation is given by a filtration of interpolation $\mathcal{F}_a^r \cup \mathcal{F}_b^s$, which is given in terms of the filtration for the forward Kolmogorov representation $\mathcal{F} = \mathcal{F}_a^t, t \in [a,b]$ which is used for prediction starting with the initial density $\phi_a\hat{\phi}_a = \mu_a$ and the filtration $\mathcal{F}_t^b$ for retrodiction for the time-reversed process with initial distribution μ_b.

We observe that q and $\hat{q}$ are in time-dependent duality with respect to the measure

$$\mu_t(x)dx = \hat{\phi}_t(x)\phi_t(x)dx, \tag{79}$$

since if we define the time-homogeneous semigroups

$$Q_{t-s}f(s,x) = \int q(s,x;t,y)f(t,y)dy, s<t \tag{80}$$
$$g\hat{Q}_{t-s}(t,y) = \int dxg(s,x)\hat{q}(s,x;t,y), s<t, \tag{81}$$

then

$$\begin{aligned}
\int dx\mu_s(x)g(s,x)Q_{t-s}f(s,x) &= \int dxg(s,x)\phi_s(x)\hat{\phi}_s(x)\frac{1}{\phi_s(x)}p(s,x;t,y)\phi_t(y)f(t,y)dy \\
&= \int dxg(s,x)\hat{\phi}_s(x)p(s,x;t,y)\frac{1}{\hat{\phi}_t(y)}f(t,x)\hat{\phi}_t(y)\phi_t(y)dy \\
&= \int dxg(s,x)\hat{q}(s,x;t,y)f(t,y)\hat{\phi}_t(y)\phi \\
&= \int dxg(s,x)\hat{Q}_{t-s}(t,y)f(t,y)\mu_t(y)dy
\end{aligned} \tag{82}$$

i.e.

$$< g, Q_{t-s}f >_{\mu_s} = < g\hat{Q}_{t-s}, f >_{\mu_t}, \quad s<t. \tag{83}$$

We shall now extend the state-space of the diffusion process to $[a,b]\times M$, to be able to transform the time-inhomogeneous processes into time-homogeneous, while the stochastic dynamics is still taken place exclusively in M. This will allow us to define the duality of the processes to be with respect to $\mu_t(x)dtdx$ and to determine the form of the exact term of the trace-torsion, and ultimately, to establish the relation between the diffusion processes and Schroedinger equations, both for potential linear and non-linear in the wave-functions. If we define time-homogeneous semigroups of the processes on $\{(t,X_t)\in[a,b]\times M\}$ by

$$P_r f(s,x) = \begin{cases} Q_{s,s+r}f(s,x), & s\geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{84}$$

and

$$\hat{P}_r g(t,y) = \begin{cases} gQ_{t-r,t}(t,y), & r\geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{85}$$

then

$$\begin{aligned}
< g, P_r f >_{\mu_t dtdx} &= \int_r^{b-r} ds < g, Q_{s,s+r}f > \mu_s = \int_b^{a+r} < g, Q_{t-r,t}f > \mu_{t-r}(x)dx \\
&= \int_{a+r}^{b} dt < g\hat{P}_{t-r}, f >_{\mu_t dx} = < \hat{P}_r g, f >_{\mu_t dtdx},
\end{aligned} \tag{86}$$

which is the duality of $\{(t,X_t)\}$ with respect to the $\mu_t dtdx$ density. Consequently, if in our space-time case we define for $a_t(x), \hat{a}_t(x)$ time-dependent one-forms on M (to be determined later)

$$B\alpha: = \frac{\partial\alpha}{\partial t} + H(g,A_t+a_t)\alpha_t \tag{87}$$
$$B^0\mu: = -\frac{\partial\mu}{\partial t} + H(g,A_t+a_t)^{\dagger}\mu_t, \tag{88}$$

and its adjoint operators

$$\hat{B}\beta = -\frac{\partial \beta}{\partial t} - H(g, -A_t + \hat{a}_t)^{\dagger}\beta_t, \tag{89}$$

$$(\hat{B})^0\mu_t = \frac{\partial \mu_t}{\partial t} - H(g, -A_t + \hat{a}_t)^{\dagger}\mu_t, \tag{90}$$

where by $H^{\dagger}$ we mean the vol_g-adjoint of the operator H defined as in eq.(22). Now

$$\begin{aligned} \int_a^b dt \int 1_{D_t}[(B\alpha_t)\beta_t] \quad - \quad & \alpha_t(\hat{B}\beta_t)]\mu_t(x)dx = \int_a^b dt \int 1_{D_t}\alpha_t\beta_t(B^0\mu_t)dx \\ - \quad & \int_a^b \int 1_{D_t}\alpha_t g([a_t + \hat{a}_t] - d\ln\mu_t, d\beta_t)\mu_t dx \end{aligned} \tag{91}$$

for arbitrary α, β smooth compact supported functions defined on $[a,b]\times M$ which we have denoted as time-dependent functions α_t, β_t, where 1_{D_t} denotes the characteristic function of the set $D_t(x) := \{(t,x) : \mu_t(x) = \phi_t(x)\hat{\phi}_t(x) > 0\}$. Therefore, the duality of space-time processes

$$< B\alpha, \beta >_{\mu_t(x)dtdx} = < \alpha, \hat{B}\beta >_{\mu_t(x)dtdx}, \tag{92}$$

is equivalent to

$$a_t(x) + \hat{a}_t(x) \quad = \quad d\ln\mu_t(x) \equiv d\ln\phi_t(x)\hat{\phi}_t(x), \tag{93}$$

$$B^0\mu_t(x) = 0, \tag{94}$$

and the latter equation being the Fokker-Planck equation for the diffusion with trace-torsion given by $a + A$, then the Fokker-Planck equation for the adjoint (time-reversed) process is valid, i.e.

$$(\hat{B})^0\mu_t(x) = 0. \tag{95}$$

Substracting eqts. $(93, 94)$ we get the final form of the duality condition

$$\frac{\partial \mu}{\partial t} + \mathrm{div_g}[(A_t + \frac{a_t - \hat{a}_t}{2})\mu_t)] \quad = \quad 0, \quad \text{for} \quad \mu_t(x) = \hat{\phi}_t(x)\phi_t(x) \tag{96}$$

Therefore, we can establish that the duality conditions of the diffusion equation in the Kolmogorov representation and its time reversed diffusion lead to the following conditions on the additional elements of the drift vectorfields:

$$a_t(x) + \hat{a}_t(x) \quad = \quad d\ln\mu_t(x) \equiv d\ln\phi_t(x)\hat{\phi}_t(x), \tag{97}$$

$$\frac{\partial \mu}{\partial t} + \mathrm{div_g}[(A_t + \frac{a_t - \hat{a}_t}{2})\mu_t] \quad = \quad 0. \tag{98}$$

If we assume that $a_t - \hat{a}_t$ is an exact one-form, i.e., there exists a time-dependent differentiable function $S(t,x) = S_t(x)$ defined on $[a,b]\times M$ such that for $t \in [a,b]$,

$$a_t - \hat{a}_t = d\ln\frac{\phi_t(x)}{\hat{\phi}_t(x)} = 2dS_t \tag{99}$$

which together with

$$a_t + \hat{a}_t = d \ln \mu_t, \tag{100}$$

implies that on $D(t,x)$ we have

$$\begin{aligned} a_t &= d \ln \phi_t, & (101) \\ \hat{a}_t &= d \ln \hat{\phi}_t. & (102) \end{aligned}$$

Remark. Note that the time-dependent function S on the 3-space manifold, is defined by eq. (98) up to addition of an arbitrary function of t, and when further below we shall take this function as defining the complex phase of the quantum Schroedinger wave, this will introduce the quantum-phase indetermination of the quantum evolution, just as we discussed already in the setting of geometry of the quantum state-space. In the other hand, this introduces as well the subject of the multivaluedness of the wave function, which by the way, leads to the Bohr-Sommerfeld quantization rules of quantum mechanics established well before it was developed as an operator theory. It is noteworthy to remark that these quantization rules, later encountered in superfluidity and superconductivity, or still in the physics of defects of condensed matter physics, are of topological character. Later we shall see that the Schroedinger wave equation contains the Navier-Stokes equations for a viscous fluid in 2D, and the probability density of the Brownian motions or still of the quantum system, will be transformed into the enstrophy of the viscous fluid obeying the Navier-Stokes equations. Thus, one might expect that Navier-Stokes equations could also have multivalued solutions, namely in the 2D case of the already established relation, the vorticity reduces to a time-dependent function.

Introduce now $R_t(x) = R(t,x) = \frac{1}{2} \ln \phi_t \hat{\phi}_t$ and $S_t(x) = S(t,x) = \frac{1}{2} \ln \frac{\phi_t}{\hat{\phi}_t}$, so that

$$\begin{aligned} a_t(x) &= d(R_t + S_t), & (103) \\ \hat{a}_(x) &= d(R_t - S_t), & (104) \end{aligned}$$

and the eq. (97) takes the form

$$\frac{\partial R}{\partial t} + \frac{1}{2} \triangle_g S_t + g(dS_t, dR_t) + g(A_t, dR_t) = 0, \tag{105}$$

where we have taken in account that $\text{div}_g A_t = 0$.

Therefore, together with the three different time-homogeneous representations $\{(t, X_t), t \in [a,b], X_t \in M\}$ of a time-inhomogeneous diffusion process $\{X_t, Q)$ on M we have three equivalent dynamical descriptions. One description, with creation and killing described by the scalar field $c(t,x)$ and the diffusion equation describing it is given by a creation-destruction potential in the trace-torsion background given by an electromagnetic potential

$$\frac{\partial p}{\partial t} + H(g, A_t)(x) p + c(t,x) p = 0; \tag{106}$$

the second description has an additional trace-torsion $a(t,x)$, a 1-form on $R \times M$

$$\frac{\partial q}{\partial t} + H(g, A + a_t) q = 0. \tag{107}$$

while the third description is the adjoint time-reversed of the first representation given by $\hat{\phi}$ satisfying the diffusion equation on the background of the reversed electromagnetic potential $-A$ in the vacuum, i.e.

$$-\frac{\partial\hat{\phi}}{\partial t}+H(g,-A_t)\hat{\phi}+c\hat{\phi}=0. \tag{108}$$

The second representation for the full trace-torsion diffusion forward in time Kolmogorov representation, we need to adopt the description in terms of the fundamental solution q of

$$\frac{\partial q}{\partial t}+H(g,A_t+a_t)q=0, \tag{109}$$

for which one must start with the initial distribution $\mu_a(x)=\hat{\phi}_a(x)\phi_a(x)$. This is a time t-irreversible representation in the "real" world, where q describes the real transition and μ_a gives the initial distribution. If in addition one traces the diffusion backwards with reversed time t, with $t\in[a,b]$ running backwards, one needs for this the final distribution $\mu_b(x)=\hat{\phi}_b(x)\phi_b(x)$ and the time t reversed probability density $\hat{q}(s,x;t,y)$ which is the fundamental solution of the equation

$$-\frac{\partial\hat{q}}{\partial t}+H(g,-A_t+\hat{a}_t)\hat{q}=0, \tag{110}$$

with additional trace-torsion one-form on $R\times M$ given by $\hat{a}$, where

$$\hat{a}_t+a_t=d\ln\mu_t(x). \tag{111}$$

where the diffusion process in the time-irreversible forward Kolmogorov representation is given by the Ito s.d.e

$$dX_t^i=\sigma_j^i(X_t)dW_t^j+(A+a)^i(t,X_t)dt, \tag{112}$$

and the backward representation for the diffusion process is given by

$$dX_t^i=\sigma_j^i(X_t)dW_t^j+(-A+\hat{a})^i(t,X_t)dt, \tag{113}$$

where $a,\hat{a}$ are given by the eqs. $(102,103)$, and $(\sigma\sigma^\dagger)^{\alpha\beta}=g^{\alpha\beta}$

We follow Schroedinger in pointing that ϕ and $\hat{\phi}$ separately satisfy the creation and killing equations, while in quantum mechanics ψ and $\bar{\psi}$ are the complex-valued counterparts of ϕ and $\hat{\phi}$, respectively, they are not arbitrary but

$$\phi\hat{\phi}=\psi\bar{\psi}. \tag{114}$$

Thus, in the following , this Born formula, once the equations for ψ are determined, will be a consequence of the constructions, and not an hypothesis on the random basis of nonrelativistic mechanics.

Therefore, the equations of motion given by the Ito s.d.e.

$$dX_t^i=(\hat{A}+\mathrm{grad}\phi)^i(t,X_t)dt+\sigma_j^i(X_t)dW_t^j, \tag{115}$$

which are equivalent to

$$\frac{\partial u}{\partial t} + H(g, A_t + a_t)u = 0 \tag{116}$$

with $a = d\ln\phi = d(R+S)$, determines the motion of the ensemble of non-relativistic particles. Note that this equivalence requires only the Laplacian for the RCW connection with the "forward" trace-torsion full one-form $Q = A + d\ln\phi = A + d(R+S)$. In distinction with Stochastic Mechanics due to Nelson, and contemporary ellaborations of this applied to astrophysics as the theory of Scale Relativity due to Nottale [55,56], we only need the form of the trace-torsion for the forward Kolmogorov representation, and this turns to be equivalent to the Schroedinger representation which interpolates in time-symmetric form between this forward process and its time dual with trace-torsion one-form given by $-A + \hat{a} = -A + d\ln\hat{\phi} = -A + d(R-S)$.

Finally, let us how this is related to the Schroedinger equation. Consider now the Schroedinger equations for the complex-valued wave function ψ and its complex conjugate $\bar{\psi}$, i.e. introducing $i = \sqrt{-1}$, we write them in the form

$$i\frac{\partial\psi}{\partial t} + H(g, iA_t)\psi - V\psi = 0 \tag{117}$$

$$-i\frac{\partial\bar{\psi}}{\partial t} + H(g, -iA_t)\bar{\psi} - V\bar{\psi} = 0, \tag{118}$$

which are identical to the usual forms. So, we have the imaginary factor appearing in the time t but also in the electromagnetic term of the RCW connection with trace-torsion given now by iA, which we confront with the diffusion equations generated by the RCW connection with trace-torsion A, i.e. the system

$$\frac{\partial\phi}{\partial t} + H(g, A_t)\phi + c\phi = 0, \tag{119}$$

$$\frac{-\partial\hat{\phi}}{\partial t} + H(g, -A_t)\hat{\phi} + c\hat{\phi} = 0, \tag{120}$$

and the diffusion equations determined by both the RCW connections with trace-torsion $A + a$ and $-A + \hat{a}$, i.e.

$$\frac{\partial q}{\partial t} + H(g, A_t + a_t)q = 0, \tag{121}$$

$$\frac{-\partial\hat{q}}{\partial t} + H(g, -A_t + \hat{a}_t)\hat{q} = 0, \tag{122}$$

which are equivalent to the single equation

$$\frac{\partial q}{\partial t} + H(g, A_t + d\ln\phi_t)q = 0. \tag{123}$$

If we introduce a complex structure on the two-dimensional real-space with coordinates (R, S), i.e. we consider

$$\psi = e^{R+iS}, \quad \psi = e^{R-iS}, \tag{124}$$

viz a viz $\phi = e^{R+s}, \hat{\phi} = e^{R-S}$, with $\psi\bar{\psi} = \phi\hat{\phi}$, then for a wave-function differentiable in t and twice-differentiable in the space variables, then, ψ satisfies the Schroedinger equation if and only if (R,S) satisfy the difference between the Fokker-Planck equations , i.e.

$$\frac{\partial R}{\partial t} + g(dS_t + A_t, dR_t) + \frac{1}{2}\triangle_g S_t = 0, \tag{125}$$

and

$$V = -\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t - A_t, dS_t). \tag{126}$$

which follows from substituting ψ in the Schroedinger equation and further dividing by ψ and taking the real part and imaginary parts, to obtain the former and latter equations, respectively.

Conversely, if we take the coordinate space given by $(\phi, \hat{\phi})$, both non-negative functions, and consider the domain $D = D(s,x) = \{(s,x) : 0 < \hat{\phi}(s,x)\phi(s,x)\} \subset [a,b] \times M$ and define $R = \frac{1}{2}\ln\phi\hat{\phi}, S = \frac{1}{2}\ln\frac{\phi}{\hat{\phi}}$, with R,S having the same differentiability properties that previously ψ, then $\phi = e^{R+S}$ satisfies in D the equation

$$\frac{\partial \phi}{\partial t} + H(g, A_t)\phi + c\phi = 0, \tag{127}$$

if and only if

$$\begin{aligned} -c &= \Big[-\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t, dS_t) - g(A_t, dS_t)\Big] \\ &+ \Big[\frac{\partial R}{\partial t} + H(g, dR_t)S_t + g(A_t, dR_t)\Big] + \Big[2\frac{\partial S}{\partial t} + g(dS_t + 2A_t, dS_t)\Big]. \end{aligned} \tag{128}$$

while $\hat{\phi} = e^{R-S}$ satisfies in D the equation

$$-\frac{\partial \phi}{\partial t} + H(g, -A_t)\hat{\phi} + c\hat{\phi} = 0, \tag{129}$$

if and only if

$$\begin{aligned} -c &= \Big[-\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t, dS_t) - g(A_t, dS_t)\Big] \\ &- \Big[\frac{\partial R}{\partial t} + H(g, dR_t)S_t + g(A_t, dR_t)\Big] + \Big[2\frac{\partial S}{\partial t} + g(dS_t + 2A_t, dS_t)\Big]. \end{aligned} \tag{130}$$

Notice that $\phi, \hat{\phi}$ can be both negative or positive. So if we define $\psi = e^{R+iS}$, it then defines in weak form the Schroedinger equation in D with

$$V = -c - 2\frac{\partial S}{\partial t} - g(dS_t, dS_t) - 2g(A_t, dS_t). \tag{131}$$

We note that from eq. (130) follows that we can choose S in a way such that either c is independent of S and thus V is a potential which is non-linear in the sense that it depends on the phase of the wave function ψ and thus the Schroedinger equation with this choice becomes non-linear dependent of ψ, or conversely, we can make the alternative choice of c depending non-linearly on S, and thus the creation-destruction of particles in the diffusion equation is non-linear, and consequently the Schroedinger equation has a potential V which does not depend on ψ .

With respect to the issue of nonlinearity of the Schroedinger equation, one could argue that the former case means that the superposition principle of quantum mechanics is broken, but then one observes that precisely due to the fact that the wave function depends on the phase, the superposition principle is invalid from the fact that we are dealing with complex-valued wave functions, and what matters, is the evolution in state-space where the complex factor has been quotiented. In the former case of a non-linear Schroedinger equation, we note that the symplectic state-space formulation is still valid [18] and the quantum geometry description incorporates non-linear quantum mechanics as is the case of the Lie-isotopic theory of Santilli, when we place in evidence in the equation, the isotopic unit of the Lie-isotopic Schroedinger-Santilli equation; see Santilli [81, 82]. In the case that V is such that the spectrum of $H(g, A+a)$ is discrete, we know already we can represent the Schroedinger equation in state-space and further study the related stochastic Schroedinger equation as described above. Finally, we have presented a construction in which by using two scalar diffusing processes $\phi, \hat{\phi}$ we have been able to subsume them into a single forward in time process with additional trace-torsion given by $a_t = d \ln \phi_t \hat{\phi}_t$, where $\mu_t = \phi_t \hat{\phi}_t$ is the distribution of the diffusion process, and obtain under eqts. (118) the Schroedinger equation (110). Alternatively, it is known that we can start with $2D$ space and the Schroedinger equation, we obtain a pair of equations, one of them being the Navier-Stokes equations for a compressible fluid where now $\phi_t \hat{\phi}_t = \psi_t \bar{\psi}_t$ equals the enstrophy of the fluid. Thus, the formal-time reversible representation can indeed be linked with the irreversible dynamics of a viscous fluid, but now the density is given by the square of the vorticity, that in this case can be associated with a function; the case for this correspondance for spatial $3D$ requires to be proved. This represents a mapping between two RCW structures (inasmuch the correspondence between the sourceless Maxwell and Dirac-Hestenes equations is another example [65]), since as was proved in [63, 67], the Navier-Stokes equations as well as the equations of passive transport of a magnetic field on a fluid, are basic examples of RCW geometries whose dynamics can be represented in terms of Brownian motions, both for boundaryless manifolds and the case of smooth boundary manifolds as well. Finally, we would like to stress that from those Brownian motions, and in particular for the cases of the Schroedinger equation and its stochastic extension in state-space, we can build Poincaré-Cartan random integral invariants [67].

Nonlinear Schroedinger equations have an important role in theoretical physics, as well as the Lie-isotopic extensions of the linear Schroedinger equation and of Quantum Mechanics, due to Santilli [81, 82, 84]. In the theory due to Santilli, it is assumed that at very short distances the quantum forces are no longer due to contact interaction representable by the quantum semigroup rules that extend the symplectic approach to nondissipative classical mechanics. These interactions arise from the overlap of the wavefunctions, and thus cannot be formally represented as in the usual approach. Thus, Santilli sets an epistemologic frontier in what is known as the interior problem of hyperdense matter and noncontact interactions, and the exterior problem which is the usually treated by to the theoretical physics to nondissipative systems. To obtain a consistent theory, a modification of the theory of numbers (known as isoarithmetic and isoalgebra) is produced incorporating an arbitrary unit (which will carry the information on the overlap of the wavefunctions of the constitutive elements of the quantum system under noncontact interaction,as well as information as the nonconstant viscosity or diffraction index, temperature, high compression), which is further carried to produce a modification of differential calculus in term of an isotopic differential, and thus a modification of the Schroedinger equation follows. In terms of an extension of the theory of general relativity, the corresponding modification is thought in terms again of the so-called interior problem corresponding to ultradense matter or spin. In the large scale exterior problem, Lie-isotopic theory recovers all the usual theories of quantum mechanics and relativity. The point of view due to Santilli is different than the one presented here, in which we present a basis for phenomenae in a form that although can be introduced in terms of scale fields, the theory is essentially topological inasmuch the torsion field is of topological origin: the nonclosure of infinitesimal parallelograms. Thus, the Schroedinger equation as presented here as well as the Brownian motions associated to RCW geometries, does not appear as linked to a particular scale, they are universal structures. Furthermore, from our analysis above, the fact that the Schroedinger equation be linear or nonlinear is not the main issue, we can always choose where to set the nonlinearity, either in the creation or destruction potential, or in the potential function V that has been historically attached to quantum physics. It is remarkable that Santilli's theory can be mapped into the present at least for certain types of units which as generators of the trace-torsion. From Santilli's theory, a new formulation of chemical bonds is produced [81, 84].

Yet, if we remain in the context of the exterior problem for quantum systems as described by quantum mechanics, in Santilli's work there is no analysis of the deeper structures and phenomenae that may arise in the exterior problem at large, nor at the relation between the aether and the exterior problem at large, as conceived in the present work, while at the level of the interior level, the existence of an elementary particle is hypothized, the so-called aetherino [85]. While in the so-called interior problem, the torsion produced by the isotopic unit which is the cornerstone of the Santilli- Lie-isotopic theory can depend on additional parameters that represent the modifications due to the overlap of the wavepackets of the quantum system and as well as due to the thermodynamics irreversible processes taking part within the boundaries set for the system to distinguish it from the canonical formalism for classical and quantum systems, the present theory presents a view of phenomenae which is free of the establishment of boundaries (which can be somehow artificial or ad-hoc). In a theory of the aether in which the non-trivial topological forces represented by geometrical torsion are at the foundations, and the structures that arise from it are valid in all scales such as vortices, spinor fields, minimal surfaces, as we shall briefly present in the next section.

Returning to the issue of the nonlinearity of the potential function V in quantum mechanics, the usual form is the known logarithmic expression $V = -b(\ln|\psi|^2)\psi$ introduced by Bialnicky-Birula and Mycielski [5]. Its importance in such diverse fields as quantum optics, superconductivity, atomic and molecular physics cannot be disregarded. Soliton solutions of nonlinear Schroedinger equations may have a role central to molecular biology, in which the DNA structure may be associated with a superconductive state. With regards as the relation between geometries, Brownian motions and the linear and Schroedinger equations, there is an alternative line of research which stems from two principles, one of them strongly related to the present one. The first is that all physical fields have to be construed in terms of scale fields starting from the fields appearing in the Einstein lambda transformations,of which, the Schroedinger wave function is an elementary example as shown above, and when further associated to the idea of a fractal spacetime, this has lead to Nottale's theory of Scale Relativity [55]. Nottales theory starts from this fractal structure to construct a covariant derivative operator in terms of the forward and backward stochastic derivatives introduced by Nelson in his theory of stochastic mechanics [52]. In Nelson's conception, Brownian motions and quantum systems are aggregates to spacetime, they are not spacetime structures themselves; this is a completely different conception that the one elaborated in this article. Working with these stochastic derivatives, the basic operator of Nottale's theory, can be written in terms of our RCW laplacian operators of the form $\frac{\partial}{\partial\tau} + H_0(i\mathcal{D}g, \mathcal{V})$ where $\mathcal{D}$ is diffusion constant (equal to $\frac{\hbar}{2m}$ in nonrelativistic quantum mechanics), and $\mathcal{V}$ is a complex differentiable velocity field, our complex drift appearing after introducing the imaginary unit $i = \sqrt{-1}$; see Nottale [55]. In the present conception, this fundamental operator in terms of which Nottale constructs his theory which has lead to numerous predictions of the positions of exoplanets confirmed by observations [56], does not require to assume that spacetime has a fractal structure a priori, from which stochastic derivatives backward and forward to express the time asymmetry construct the dynamics of fields. We rather assume that at a fundamental scale which is generally associated with the Planck scale, we can represent spacetime as a continuous in which what really matters are the defects in these continuous, and thus torsion has such a fundamental role. The fractal structure of spacetime arises from the association between the RCW laplacian operators which as we said coincide with Nottale's covariant derivative operator, and the Brownian motions which alternatively, can be seen as constructing the spacetime geometry. So there is no place as to the discussion of what goes first, at least in the conception in the present work. The flow of these Brownian motions under general analytical conditions, define for every trial Wiener path, an active diffeomorphism of spacetime. But this primeval role of the Brownian motions and fractal structures, stems from our making the choice -arbitrary, inasmuch as the other choice is arbitrary- as the fundamental structure instead of choosing the assumption of having a RCW covariant derivative with a trace-torsion field defined on a continuous model of spacetime. In some sense the primeval character of Brownian motions as a starting point is very interesting in regards that they can be constructed as continuous limits of discrete jumps, as every basic book in probability presents [18], and thus instead of positing a continuous spacetime, we can think from the very beginning in a discrete spacetime, and construct a theory of physics in these terms as suggested in [63] [12] In this case, instead of working with the field of the real number or its complex or biquaternion extensions, one can take a p-adic field, such as the one defined by the Mersenne prime number $2^{127} - 1$ which is approximately equal to the square of the ratio between the Planck mass and the proton mass. This program and its relations with the fundamental constants of physics, was elaborated independently by a number of authors and an excellent presentation can be found in Castro [9] and references therein.In fact, a theory of physics in terms of discrete structures associated to the Mersenne prime numbers hierarchy, has been constructed in a program

[12] Prof. Shan Gao,has initiated a program of construction of quantum mechanics as random discontinuous motions in discrete spacetime, in his recent work *Quantum Motion, Unveiling the Mysterious Quantum World*, Arima Publ., Suffolk (U.K.), 2006.

developed by P. Noyes, T. Bastin, P. Kilmister and others; see [57].

10.1. The Extension to The Many-body Case

Up to know we have presented the case of the Schroedinger equation for an ensemble of one-particle systems on space-time. Of course, our previous constructions are also valid for the case of an ensemble of interacting multiparticle systems, so that the dimension of the configuration space is $3d+1$, for indistinguishable d particles; the general case follows with minor alterations. If we start by constructing the theory as we did for an ensemble of one-particle systems (Schroedinger's cloud of electrons), we can still extend trivially to the general case, by considering a diffusion in the product configuration manifold with coordinates $X_t = (X_t^1, \ldots, X^d) \in M^d$, where M^d is the d Cartesian product of three dimensional space with coordinates $X_t^i = (x_t^{1,i}, x_t^{2,i}, x_t^{3,i}) \in M$, for all $i = 1, \ldots, d$. The distribution of this is $\mu_t = E_Q \circ X_t^{-1}$, which is a probability density in M^d. To obtain the distribution of the system on the three-dimensional space M, we need the distribution of the system X_t:

$$U_t^x := \frac{1}{d} \sum_{i=1}^{d} \delta_{x_i}. \tag{132}$$

which is the same as

$$U_t^x(B) = \frac{1}{d} \sum_{i=1}^{d} 1_B(X_t^i), \tag{133}$$

where $1_B(X_t^i)$ is the characteristic system for a measurable set B , equal to 1 if $X_t^i \in B$, for any $i = 1 \ldots, d$ and 0 otherwise. Then, the probability density for the interacting ensembles is given by

$$\mu_t^x(B) = E_Q[U_t^x(B)], \tag{134}$$

where E_Q is the mean taken with respect to the forward Kolmogorov representation presented above, is the probability distribution in the three-dimensional space. Therefore, the geometrical-stochastic representation in actual space is constructable for a system of interacting ensembles of particles. Thus the criticism to the Schroedinger equation by the Copenhagen school, as to the unphysical character of the wave function since it was originally defined on a multiple-dimensional configuration space of interacting system of ensembles, is invalid.

11. The Navier-Stokes Equations and Riemann-Cartan-Weyl Diffusions

We have seen that quantum mechanics is an example of spacetime structures of RCW. We have shortly discussed the fact that the Navier-Stokes equations for viscous fluids are another example of this. In this section we shall present the proofs of this statements.

In the sequel, M is a compact orientable (*without boundary*) n-manifold with a Riemannian metric g. We provide M with a 1-form $u(\tau, x) = u_\tau(x)$ satisfying the invariant Navier-Stokes equations (NS in the following) ,

$$\frac{\partial u}{\partial \tau} + P[\nabla^g_{\hat{u}_\tau} u_\tau] - \nu \triangle_1 u_\tau = 0, \tag{135}$$

where P is the projection operator to the co-closed term in the de Rham-Kodaira-Hodge decomposition of 1-forms. We have proved in [67], that we can rewrite NS in the form of a non-linear diffusion equation[13]

$$\frac{\partial u}{\partial \tau} = PH_1\Big(2\nu g, \frac{-1}{2\nu}u_\tau\Big)u_\tau, \tag{136}$$

which means that NS for the velocity of an incompressible fluid is a a non-linear diffusion process determined by a RCW connection. This connection has $2\nu g$ for the metric, and the time-dependant trace-torsion of this connection is $-u/2\nu$. Then, the drift of this process does not depend explicitly on ν, as it coincides with the vectorfield associated via g to $-u_\tau$, i.e.$-\hat{u}_\tau$. Let us introduce the vorticity two-form

$$\Omega_\tau = du_\tau, \quad \tau \geq 0. \tag{137}$$

Now, apply d to eq. (132); since $d\triangle_1 u_\tau = \triangle_2 du_\tau = \triangle_2\Omega_\tau$ and $dL_{\hat{u}_\tau} = L_{\hat{u}_\tau}du_\tau = L_{\hat{u}_\tau}\Omega_\tau$ we obtain the evolution equation for the vorticity (the so called Navier-Stokes equation for the vorticity):

$$\frac{\partial \Omega_\tau}{\partial \tau} = H_2\Big(2\nu g, \frac{-1}{2\nu}u_\tau\Big)\Omega_\tau. \tag{138}$$

Now, if we know Ω_τ for any $\tau \geq 0$, we can obtain u_τ by inverting the definition (133). Namely, applying δ to (133) we obtain the Poisson-de Rham equation

$$H_1(g,0)u_\tau = -d\delta u_\tau - \delta\Omega_\tau, \quad \tau \geq 0. \tag{139}$$

Thus, the vorticity Ω_τ is a source for the velocity one-form u_τ, for all τ together with the predetermined expression for δu_τ; in the case that M is a compact euclidean domain, equation (60) is integrated to give the Biot-Savart law of Fluid Mechanics. If furthermore the fluid is incompressible, i.e. $\delta u_\tau = 0$, then we get the Poisson-de Rham equation for the velocity having the vorticity as a source,

$$H_1(g,0)u_\tau = -\delta\Omega_\tau, \quad \tau \geq 0. \tag{140}$$

In $3D$ this is none other that the Biot-Savart law but applied to fluid dynamics, instead of electromagnetism.

Theorem. *Given a compact orientable Riemannian manifold with metric g, the Navier-Stokes equation* (132) *for fluid with velocity one-form* $u = u(\tau, x)$*, assuming sufficiently regular conditions, are equivalent to a diffusion equation for the vorticity given by* (132) *with* u_τ *satisfying the Poisson-de Rham eq.* (135) *for the compressible case and eq.* (136) *for the incompressible one. The RCW connection on M generating this process is determined by the metric* $2\nu g$ *and a trace-torsion* 1*-form given by* $-u/2\nu$.

[13]While in the boundaryless case P commutes with $\triangle_1$, in the case of M with smooth boundary this is no longer true so that we have to take $P\triangle_1 u_\tau$ instead of the viscosity term in eq. (131), and we are left with the non-linear diffusion equation (132) in any case.

Observations. This characterization of NS in terms of a gauge structure, will determine all the random representations for NS which we shall present in this article. We would like to recall that in the gauge theory of gravitation (see Blagojevic [14] the torsion is related to the translational degrees of freedom present in the Poincaré group, i.e. to the gauging of momentum. Here we find a similar, yet dynamical situation, in which the trace-torsion is related to the velocity and the angular momentum is derived from it simply by considering the vorticity of the fluid. We conclude this chapter noting that with this constructions we can finally give the most general analytical representations for the Navier-Stokes equations using the Brownian motions corresponding to the Navier-Stokes operator for manifolds with and without boundary and in Euclidean domains and semidomains [63, 67].

12. Turbulence and the Riemann-Cartan-Weyl Torsion

Turbulence is a universal phenomenae inasmuch viscous fluids are universal. In particular, the role of turbulence in astrophysics has been discussed by several authors [4, 95] . Evidence of turbulence for the origin of galaxy formation has been detected by observations [95]. Gibson has extensively discussed the formation of the gravitational field, galaxies and the Universe from a turbulent fluid [23] and contrasted with advantage the usual approach through the Jeans law. In the present approach in which viscous fluids, gravitational fields, quantum mechanics are all instances of a single geometrical structure and its random counterpart, this seems extremely natural.

In this section we want to introduce a treatment of turbulence which is independent of the particular equations of dynamics and is directly associated with the RCW geometries through the structure of the trace-torsion one-form, Q, whose conjugate vectorfield, whenever the metric is Minkowski or in an arbitrary Riemannian (i.e. positive-definite) metric is established from the beginning,or still, in the latter case, whenever we have a noise tensor which generates the Riemannian metric through the eq. (15). The clue to this is through the ideas elaborated by R. Kiehn, the topological (also called, the Pfaffian dimension) dimension of Q. So we consider the set of differential forms on 4-dimensional spacetime given by

$$\{Q, dQ, Q \wedge dQ, dQ \wedge dQ\}, \tag{141}$$

which cannot have higher degree differential forms since $d(dQ) = 0$ whenever the coefficient functions of Q are twice differentiable. Then, we follow Kiehn by recalling that the topological dimension of Q is the minimal number of coordinates in M on which Q depend. Thus, if $dQ = 0$, then in a connected neighborhood of M, we can find a differentiable function, say f, such that $Q = df$, i.e. Q is an exact form in that neighbourhood. Trivially $dQ = 0$ as well as the higher degree forms of the above set. In this case, it is clear that Q can be parametrized by a one-dimensional set given by the inverse image by f of all its values in the real line, and thus for an exact one-form the topological dimension is equal to 1. Let us consider the case that Q is not exact and furthermore $Q \wedge dQ = 0$. By the well known Frobenius integrability theorem, then Q has topological dimension equal to 2, i.e. M can be at least locally foliated by two-dimensional submanifolds on which Q is defined; this corresponds to a reversible dynamics given by the integral flow of Q that lies in this two-dimensional submanifold. Now assume in the contrary that $Q \wedge dQ \neq 0$ and furthermore $dQ \wedge dQ \neq 0$, so that being this a top degree differential form on M, in this case Q has topological dimension equal to 4, and thus equal to the dimension of spacetime. In this case, the integral flow of the drift vectorfield $\hat{Q}$, for a positive-definite metric, lies in a four dimensional submanifold. Otherwise, if $dQ \wedge dQ$, then Q has topological dimension equal to 3; in this case, the drift vectorfield has a flow lying in a three-dimensional submanifold.

In the case of topological dimension equal to 4, we extend Kiehn [33] defining a vectorfield called the topological torsion by the rule

$$Q \wedge dQ = i_T \mathrm{vol_g}. \tag{142}$$

If we introduce the Hodge star operator $*$ defined by g, we have that if T^g denotes the one-form given by the g conjugate of the vectorfield T (i.e. $T^g = a_\alpha dx^\alpha$ with $a_\alpha = g_{\alpha\beta} T^\beta$, where $T = T^\beta \frac{\partial}{\partial x^\beta}$ is the coordinate expression for T, then [22]

$$*T^g = i_T \mathrm{vol_g} = Q \wedge dQ, \tag{143}$$

which is Kiehn's topological torsion three-form obtained by duality from T^g. When g is the Euclidean metric we retrieve the original definitions [33]. Although the present formulation retrieves the trivial metric case, it is more general since it includes the noise tensor of the Brownian motions having the drift vectorfield produced by the g-conjugate of Q, producing the metric by eq. (15), so in spite the exterior differential operator d in terms of which define the topological dimension is independent of the background noise, the topological torsion one-form and the topological torsion vectorfield here introduced, do depend on the metric (and the background noise) through the relation (138) and (139). So the physical meaning of this terms incorporates the background noise tensor, contrarily to Kiehn's approach in which the topological approach is unlinked to noise. In this respect, the presentation here introduced has incorporated the dynamics of the vacuum while in the approach due to Kiehn, the vacuum is absent altogether in the definitions. As it stands, the present constructions are associated to the mean motion of the Brownian motions through their drift.

We must remark that altough the present constructions apply to an arbitrary spacetime trace-torsion one-form, and thus includes the case of a three-dimensional fluid velocity $u_\tau(x)$ which is also is time-dependent and obeys Navier-Stokes equations, the present exterior differential has an additional time derivative component which is missing in the exterior differential that we encountered when introducing the Navier-Stokes equations. Indeed, when there we wrote du_τ this time derivative is absent. The presentation we are giving of the topological dimension, incorporates time as an active parameter for its definition. This is very important, since as we shall see, the topological dimension is related to coherent structures, turbulence, chaos, etc., in a formalization in which statistical considerations are absent completely. In our approach that differs from the one due to Kiehn, does *not* mean that this structures are not related to an intrinsic randomness, such as the case of the Brownian motions generated by the RCW geometries which have trace-torsion given by the one-form Q generating through the g-conjugate -where g arises from noise- the drift. In this respect, the topological dimension incorporating this active time parameter, coincides with Pensinger and Paine's idea of an active time operator in their study of turbulence in atmospheric fluids which is none-other than the exterior differential in 4D written in a quaternionic base; see [60]. We recall that Kozyrev's conception of time is exactly that of an active operator, as we have already discussed above, so what we are actually doing here is presenting a topological theory of structures and further below, of processes, in which time is an active operator for their formation and preservation. Furthermore, these constructions can be related to a multivalued logic with a time operator that arise from associating the primitive distinction in the calculus of distinctions due to Spencer-Brown [75] which is based in a nilpotence condition similar that the condition $d^2 = 0$ of the differential operator; we shall return to this issue elsewhere.

We now compute the four-form $dQ \wedge dQ$ to obtain

$$dQ \wedge dQ = di_T \mathrm{vol_g} = L_T \mathrm{vol_g} - i_T d\mathrm{vol_g} = L_T \mathrm{vol_g}, \tag{144}$$

which is still equal to $\Gamma \mathrm{vol_g}$ with $\Gamma = \mathrm{div_g} T$ (by definition of the divergence, see eq. (4.28) in [22] and therefore

$$dQ \wedge dQ = \Gamma \mathrm{vol_g} \equiv \mathrm{div_g}(T) \mathrm{vol_g}. \tag{145}$$

Thus, Γ is the topological dissipation function. It expresses how the 4-volume defined by the 4-form $dQ \wedge dQ$ shrinks or expands in terms of the Riemannian volume $\mathrm{vol_g}$. In fact, this 4-form is the Liouville form produced by the symplectic 2-form dQ, so that here spacetime adquires a symplectic structure, i.e. a nondegenerate closed 2-form on four dimensional spacetime. In a same domain of M we can actually have different topologies in the sense of Pfaff. We note whenever the topological dimension coincides with the spacetime dimension 4, topological torsion is related to a system whose evolution occupies the 4-dimensional domain, with the possibility that whenever in this domain T is divergenceless, then the topological dimension of the trace-torsion Q [14] collapses to 3, thus we have a contact Hamiltonian reversibles structure defined by $Q \wedge dQ$, corresponding to spacetime defects which are nonequilibrium long lived closed systems, generically spacetime dislocations, or still coherent or stationary structures such as vortices, solitons, dislocations, minimal surfaces, etc. The domains on which the topological dimension of Q is 4 correspond to thermodinamically open irreversible systems, and in the direction of T, evolution is irreversible; according to Kiehn, these dynamics correspond to turbulent systems, in our case, associated to the trace-torsion Q whose conjugate vectorfield is the drift of the Brownian motions. In the case we have topological dimension equal to 2 or 1, this corresponds to isolated systems in equilibrium. We we would like to remark that we can still follow Kiehn presenting a theory of systems based upon the action of vectorfields on the trace-torsion Q, which would then correspond to the evolution of arbitrary processes on the background of the RCW Brownian motions. This description should be elaborated to establish a topological-geometrical approach to the processes in interaction with a universal field, on which we have the action of an active time operator, described by Kozyrev [37–39], or still the geophysical, ionspheric and solar processes described by Korotaev, Serdyuk and Gorohov [36].

[14] It should not be confused with Kiehn's notation for the heat one-form which in this formalism coincides with $L_V Q$ for V a spacetime vectorfield which is thought as a process acting on the system defined by Q (noted A in [33].

13. Introduction to the Lie-Santilli Mathematical Isotopies

In this section we shall present briefly the mathematical core of Hadronic Mechanics [81,84] as a preliminary to the introduction of this theory as that of the strong interactions proposed by Santilli. This theory requires the introduction of the so-called isotopies of the number fields, functions, Lie groups, manifolds, laplacians, Hilbert spaces, to produce the modification of the Schroedinger equation called the iso-Scroedinger equation introduced by the former isotopies which require fundamentally the introduction of a generalized unit (instead of the usual numerical unit, 1) that accounts for the strong interactions. The introduction of the isotopies are demanded by consistency with the action of non-unitary transformations. [15]The ensuing theory, Hadronic Mechanics (HM), together with the modifications of special relativity and general relativity produced by the introduction of the generalized so-called isotopic lift, is claimed by Santilli to be a unified theory for physics that stems from acknowledging the special conditions and phenomenology of the strong interactions as well as those of inhomogeneous media instead of the particular case of the Lorentz invariant vacuum. While QM stemmed from a theory for quantum physics on phenomenae occuring on such a vacuum deemed to be unrelated to spacetime geometries, the case of the strong interactions fells short of occuring on such conditions, yet those on a very inhomogeneous spacetime with overlapping wave functions, so a nonlinear theory is necessary, yet this requires a theory in which the spacetime geometry for such interactions and more generally as a foundational problem for all physics. Furthermore, in placing the foundations of all the modifications on the new isotopic unit, while refraining from searching for a deeper understanding of QM in particular of its relations with spacetime introduced by torsion, it contains in essence a shallow paradigm which stops short of acknowledging the torsion structure of spacetime as a basis for its constitution, as well as that of subjectivity (in particular logic) and the fusion of it with the physical realm as elaborated by Rapoport [75, 77]. In this section we shall introduce HM in terms of the isotopic modifications introduced by Santilli, yet also in terms of torsion as presented above.

Thus we shall introduce very briefly the theory by introducing the Lie-Santilli-isotopic unit; more comprehensive treatment is available in [84] to which we refer for further details. The prescription is to introduce an arbitrary non-unitary operator U and to substitute the unit I by the isotopic unit

$$\hat{I} = U \times I \times U^{\dagger} \neq I. \tag{146}$$

where we have denoted the multiplication by $\times$ instead of the usual juxtaposition notation

[15]Santilli claims that new mathematics are demanded for the study of not only the strong interactions, meaning by this that new formalisms, as provided by the previous isotopies or still more general hyperstructures [84], are required for physics, chemistry and biology. Unfortunately this discussion, due to length restrictions, cannot be detailed thoroughly as a subject of such importance merits, yet it is the understanding of this author that this claim might be misconceived if related to (exclusively, for Santilli) formalisms: while the strong interactions may require the isotopies, it is not a general rule that new formalism is required in the sense stated by Santilli, very much it is the case that instead of formalisms, interpretations of known mathematical structures are required to shed a completely new paradigm of physics and science as a whole [75, 77]. This new interpretation stems from torsion fields, time gestalts and a time operator, multivalued logics and most essentially, the Klein bottle surface. It is the latter that plays the role of (isotopic, or whatever) unit, and the hyper Klein bottle that gives a family of units placed in heterarchical relation.

for the product, so that

$$\hat{T} = (U \times I \times U^{\dagger})^{-1} = \hat{T}^{\dagger}. \tag{147}$$

The usual Hilbert space of quantum mechanics, is denoted by $\mathcal{H} = \{|\Phi>, |\Psi>: <\Phi|\Psi> \in C(c,+,\times), <\Psi,\Psi>=1\}$, where $C(c,+,\times)$ denotes the field of complex numbers with the usual addition and multiplication. The evolution equation in the Santilli-Lie-isotopic theory of an observable is given by the equation

$$i\frac{dA}{dt} = [A\hat{;}H] := A \times T \times H - H \times T \times A, \tag{148}$$

so that

$$A(t) = e^{i\times H\times T\times t} \times A(0) \times e^{-i\times t\times T\times H}. \tag{149}$$

The problem with this quantum evolution is that it is non-unitary over the Hilbert space $\mathcal{H}$ over the field $C(c,+,\times)$. We have the following fundamental result, known as the López Lemma [83].

Theorem 1. *All possible non-unitary deformations of QM computed on a conventional Hilbert space $\mathcal{H}$ over the field $C(c,+,\times)$ have the following aspects:*

(i) *Lack of invariance of the unit, and consequently the lack of applicability to measurements.*

(ii) *Lack of preservation of the Hermiticity in time, and consequently the lack of unambiguous observables.*

(iii) *Lack of invariant eigenfunctions and their transforms, and consequently the lack of invariant numerical predictions.*

The general situation of non-unitary deformations computed on general Hilbert spaces will be addressed below. As a corollary of the Theorem 1, on $(\mathcal{H}, C(c,+,\times))$ non-unitary quantum deformations do not give invariant probabilities, nor posses unique invariant physical laws. While in QM unitary time evolution implies causality, in its non-unitary deformations there is a violation of causality.

So let us proceed to present the solution to this problem provided by Santilli, the construction of a non-unitary image of QM. Firstly, the product in the generalized enveloping algebra $\hat{\xi}$ is given by elements of the form $U \times A \times B \times U^{\dagger} = \hat{A} \times \hat{T} \times \hat{B} := \hat{A}\hat{\times}\hat{B}$ for $\hat{A} = U \times A \times U^{\dagger}$ and $B = U \times B \times B^{\dagger}$. For a Hilbert space $(\mathcal{H}, <>, C(c,+,\times))$ we introduce the Lie-Santilli isotopic Hilbert space $\hat{\mathcal{H}}$ of elements of the form $|\hat{\psi}> = U \times |\psi>$ and $<\hat{\phi}| = <\phi| \times U^{\dagger}$, with inner product given by transforming the original $\mathcal{H}$ inner product by the non-unitary transformation

$$<\Phi,\Psi> \to <\Phi| \times U^{\dagger} \times U^{\dagger -1} \times U^{-1} \times U|\Psi> = <\hat{\Phi}| \times \hat{T} \times |\Psi> \equiv <\hat{\Phi}|\hat{\times}|\hat{\Psi}>. \tag{150}$$

The generalized enveloping algebra $\hat{\xi}$ is still associative

$$(\hat{A}\hat{\times}\hat{B})\hat{\times}\hat{C} = \hat{A}\hat{\times}(\hat{B}\hat{\times}\hat{C}), \tag{151}$$

with identity given by $\hat{I}$, since $\hat{I}\hat{\times}\hat{A} = \hat{A}\hat{\times}\hat{I} = \hat{A}$. The modified Lie algebra is given by $[\hat{A}\hat{;}\hat{B}] = \hat{A} \times T \times \hat{B} - \hat{B} \times T \times \hat{A}$, which is isomorphic to the original one if $\hat{I}$ is positive-definite.

Now let us see how the problem of hermiticity in the non-unitary frame is obtained. We have,

$$<\Psi| \times \hat{T} \times (H \times \hat{T}) \times |\Psi> = <(\hat{\Psi}| \times \hat{T} \times H^{\dagger}) \times |\hat{\Psi}>, \tag{152}$$

which yields

$$H^{\hat{\times}\dagger} = \hat{T}^{-1} \times \hat{T} \times H^{\dagger} \times \hat{T} \times T^{-1}. \tag{153}$$

Thus, starting with an hermitean operator H at $t = 0$, then $\hat{H} = U \times H \times U^{\dagger}$ remains hermitean under non-unitary transformations. But we note that the hermiticity is not computed in $(\mathcal{H}; <|>, C(c,+,\times))$ but in $(\hat{\mathcal{H}}, <|\hat{\times}|>, \hat{C}(\hat{c},\hat{+},\hat{\times}))$, where $\hat{C}(,\hat{c},+,\hat{\times})$ is the Santilli-Lie isotopic lift of $C(c,+,\times)$ with elements of the form $\hat{c} = c \times \hat{I}$, where $\hat{I}$ not necessarily belongs to $\hat{C}$; this isofield is defined by the isosum $\hat{c}_1 + \hat{c}_2 = (c_1 + c_2) \times \hat{I}$ and the isoproduct $\hat{c}_1 \hat{\times} \hat{c}_2 = \hat{c}_1 \times \hat{T} \times \hat{c}_2 = (c_1 \times c_2) \times \hat{I}$. Then $\hat{I} = \hat{T}^{-1}$ is the left and right multiplicative unit in $\hat{C}(\hat{c},+,\hat{\times})$. We further have that $\hat{0} = 0$ satisfies $\hat{c} + \hat{0} = \hat{c}$. Furthermore, $\hat{c}^2 = \hat{c} \hat{\times} \hat{c} = \hat{c} \times \hat{T} \times \hat{c}$, $\hat{c}^{\hat{\frac{1}{2}}} = c^{\frac{1}{2}} \times \hat{I}^{\frac{1}{2}}$. The quotient is defined by $\hat{a}/^{\hat{\times}}\hat{b} = (\frac{a}{b} \times \hat{I})$, and $|\hat{c}| = |c| \times \hat{I}$ and finally for an arbitrary Q, $\hat{c} \hat{\times} Q = c \times \hat{I} \times T \times Q = c \times I \times Q = c \times Q$. These isofields can be very elegantly introduced by the expedience of applying the non-unitary transforms to the classical fields, which are the basis of this theory; see [31, 84].

The modular action $\hat{H} \hat{\times} |\hat{\Psi}> = \hat{H} \times \hat{T} \times |\hat{\psi}>$ with $<\hat{\psi}|\hat{\times}|\psi> = <\hat{\psi}| \times T \times |\hat{\psi}>$ has for generalized unit $\hat{I} = \hat{T}^{-1}$, because it is the only object such that $\hat{I} \hat{\times} |\hat{\psi}> = |\hat{\psi}>$. Consequently, referral to $C(c,+,\times)$ with unit I is inconsistent, and then $\hat{\mathcal{H}}$ must be referred to $\hat{C}$ with basic unit $\hat{I}$. Then, to achieve consistency, the Santilli-iso-Hilbert invariant iso-inner product is defined by

$$<\hat{\Phi}|\hat{\Psi}>^{\hat{I}} := <\hat{\Phi}|\hat{\times}|\hat{\Psi}> \times \hat{I} = <\hat{\Phi}| \times \hat{T} \times |\hat{\Psi}> \times \hat{T}^{-1} \in \hat{C}, \tag{154}$$

with normalization $<\hat{\Phi}|\hat{\times}|\hat{\Phi}> = I$. Note that from eqs. $(153, 154)$ follows that isohermiticity coincides with conventional Hermiticity. As a result, all conventional quantum mechanical observables are preserved for the above iso-Hilbert spaces over isofields; for the details see [84]. It is important to remark that the transformation that carries $H \times p|\psi>$ to $\hat{H} \hat{\times} |\hat{\psi}>$ satisfies linearity on isospace over isofields. The recovery of linearity in isospace is achieved by the embedding of the nonlinear terms in the isounit. Furthermore, any nonlinear theory with a Hamiltonian operator $H(p,x,\psi,\ldots)$ can always be rewritten by factorizing the nonlinear terms, which can then be assumed as the isotopic element of the theory. Indeed, if we have

$$H(p,x,\psi,\ldots) \times |\hat{\psi}> = H_0(x,p) \times \hat{T}(x,p,\psi,\ldots) \times |\hat{\psi}> := \hat{H}_0(,p) \hat{\times} |\hat{\psi}>,$$

with $\hat{T} := H_0^{-1} \times H$, with H_0 being the maximal hermitean operator, representing the total energy. Thus, the theory presented above of the relations between diffusions, spacetime geometries and nonrelativistic QM, in the case of non-linear Schroedinger equations can be framed equivalently in terms of an isoHilbert theory.

It is about time to present a general class of generalized units that appear in HM for the characterization of the strong interactions. Namely,

$$\hat{I} = \mathrm{diag}(n_1^2, n_2^2, n_3^2, n_4^2) \times \exp(tN\Big(\frac{\psi_{\uparrow}}{\hat{\psi}_{\downarrow}} + \frac{\partial \psi_{\downarrow}}{\partial \hat{\psi}_{\downarrow}} + \ldots\Big) \times \int d^3x \psi_{\uparrow}^{\dagger}(x) \psi_{\downarrow}(x), \tag{155}$$

where the quantities n_1^2, n_2^2, n_3^2 represent the extended, non-spherical deformable shapes of the hadron, n_4^2 its density, the quantities $\frac{\psi_\uparrow}{\hat{\psi}_\downarrow} + \frac{\partial\psi_\downarrow}{\partial\hat{\psi}_\downarrow} + \ldots$ represent a typical non-linearity, and the integral in the exponent, represents a typical non-linearity due to the interpenetration and overlapping of the charge distributions. Notably a coupling of spin-up and spin-down particles is present in the generalized unit. Whenever the hadrons are perfectly spherical and rigid, then we can take the density $n_4^2 = 1$ and the parameters of deformations can also be set equal to 1; if furthermore, their distances is such as to be nor interpenetration, then the integrand is zero and the exponential term is equal to 1; thus, $\hat{I} = I$, and we are in the situation of QM, where the unit is given in terms of the torsion structure constants of the Lie algebra, and dynamically, from the gradient logarithm of the wave function. The present choice of the isotopic unit has lead to the first ever model of a Cooper model with explicit attractive force between the pair of identical electrons with excellent agreement with experimental data [3]. As closing remark we note that there is no general rule for the actual construction of the isotopic unit. which is ad-hoc to the particular phenomenology, as the above example shows. The drawback for this is that the pledged universality is such that there is no general rule for the choice of the isotopic unit; it is essentially ad-hoc.

13.1. Santilli-Lie Isotopies of the Differential Calculus and Metric Structures, and the Iso-Schroedinger Equation

To present the iso-Schroedinger equation, we need the isotopic differential calculus and the isotopic lift of manifolds, the so-called isomanifolds, due to Tsagas and Sourlas [31, 92]; we shall follow here the above notations. We start by considering the manifold M to be a vector space with local coordinates, which for simplicity we shall from now fix them to be a *contravariant* system, $x = (x^i), i = 1, \ldots, n$, unit given by $I = \mathrm{diag}(1, \ldots, 1)$ and metric g which we assumed diagonalized. We shall lift this structure to a vector space $\hat{M}$ provided with isocoordinates $\hat{x}$, isometric $\hat{G}$ and defined on the isonumber field $\hat{F}$, where F can be the real or complex numbers; we denote this isospace by $\hat{M}(\hat{x}, \hat{G}, \hat{F})$. The isocoordinates are introduced by the transformation $x \mapsto U \times x \times U^\dagger = x \times \hat{I} := \hat{x}$. To introduce the *contravariant* isometric $\hat{G}$ we start by considering the transformation [16]

$$g \mapsto U \times g \times U^\dagger = \hat{I} \times g := \hat{g}. \tag{156}$$

Yet from the Definition 3.2.3 of vol. III in [84] follows that the isometric is more properly defined by $\hat{G} = \hat{g} \times \hat{I}$. Thus we have a transformed $M(x, g, F)$ into the isospace $\hat{M}(\hat{x}, \hat{G}, \hat{F})$. Thus the projection on $M(x, g, F)$ of the isometric in $\hat{M}(\hat{x}, \hat{G}, \hat{F})$ is defined by a contravariant tensor, $\hat{g} = (\hat{g}^{ij})$ with components

$$\hat{g}^{ij} = (\hat{I} \times g)^{ij}. \tag{157}$$

If we take $\hat{I} = \psi^2(x) \times I$ we then retrieve the Weyl scale transformations, with ψ a scale field depending only on the coordinates of M which we encountered already. If we start with g being the Euclidean or Minkowski metrics, we obtain the iso-Euclidean and iso-Minkowski metrics; in the case we start with a general metric as in GR, we obtain Isorelativity. We

[16] We shall assume, as usual, a diagonal metric.

shall now proceed to identify the isotopic lift of the noise tensor σ which verifies eq. (9), i.e. $\sigma \times \sigma^{\dagger} = g$. The non-unitary transform of (a diagonalized) σ is given by

$$\sigma \mapsto U \times \sigma \times U^{\dagger} = \sigma \times \hat{I} := \hat{\sigma}. \tag{158}$$

Then,

$$\hat{\sigma} \hat{\times} \hat{\sigma} = \sigma \times \hat{I} \times \hat{T} \times (\sigma \times \hat{I})^{\dagger} = (\sigma \times \sigma^{\dagger}) \times \hat{I} = g \times \hat{I} = \hat{g}. \tag{159}$$

Thus the isotopic lift of the noise tensor defined on $\hat{M}(\hat{x}, \hat{G}, \hat{R})$ is given by $\hat{\sigma} = \sigma \times \hat{I}$ which on projection to $M(\hat{x}, \hat{G}, R)$ we retrieve σ. We know follow the notations and definitions of Section 3.2.5 for the isotopic differential, and for isofunctions. We introduce the isotopic gradient operator of the isometric $\hat{G}$ (the $\hat{G}$-gradient, for short), $\widehat{\mathrm{grad}_{\hat{G}}}$ applied to the isotopic lift $\hat{f}(\hat{x})$ of a function $f(x)$ is defined by

$$\widehat{\mathrm{grad}_{\hat{G}}} \hat{f}(\hat{x})(\hat{v}) = \hat{G}(\hat{d}\hat{f}(\hat{x}) \hat{;} \hat{v}), \tag{160}$$

for any vector field $\hat{v} \in T_{\hat{x}}(\hat{M})$, $\hat{x} \in \hat{M}$; we have denoted the inner product as $\hat{;}$ to stress that the inner product is taken with respect to the product in $\hat{F}$. Hence, the operator $\widehat{\mathrm{grad}_{\hat{G}}} \hat{f}(\hat{x})$ can be thought as the isovector field on the tangent manifold to $\hat{M}(\hat{x}, \hat{G}, \hat{F})$ defined by

$$\hat{G}^{\alpha\beta} \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\beta}} = \hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{i}} \times \hat{I}. \tag{161}$$

Therefore, the projection on $\hat{M}(\hat{x}, \hat{g}, F)$ of the $\hat{G}$-gradient vector field of $\hat{f}(\hat{x})$ is the vector field with components

$$\hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}} = \hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}}. \tag{162}$$

This will be of importance for the determination of the drift vector field of the diffusion linked with the Santilli-iso-Schroedinger equation. We finally define the isolaplacian as

$$\hat{\triangle}_{\hat{g}} = \hat{g}^{\alpha\beta} \hat{\times} \hat{D}_{\frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\alpha}}} \hat{\times} \hat{D}_{\frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\beta}}} \tag{163}$$

Here $\hat{D}_{\frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\alpha}}}$ is defined accordingly with Definition 3.2.13 in [84],

$$\hat{D}_{\frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\alpha}}} \hat{X}^{\beta} = \frac{\hat{\partial} \hat{X}^{\beta}}{\hat{\partial} \hat{x}^{\beta}} + \widehat{\begin{Bmatrix} \beta \\ \gamma\alpha \end{Bmatrix}} \hat{\times} \hat{X}^{\gamma}, \tag{164}$$

and hence it is the isocovariant differential with respect to the Levi-Civita isoconnection with isoChristoffel coefficients

$$\widehat{\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}} = \frac{\hat{1}}{\hat{2}} \Big(\frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\beta}} \hat{g}_{\nu\gamma} + \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\gamma}} \hat{g}_{\beta\nu} - \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\nu}} \hat{g}_{\beta\gamma} \Big) \hat{\times} \hat{g}^{\alpha\nu}. \tag{165}$$

We remark that from Observations 1 follows that alternatively we can define the more simpler laplacian by taking instead

$$\hat{\triangle}_{\hat{g}} = \hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\alpha}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\beta}}. \tag{166}$$

In both cases we take $\hat{\sigma}$ for the corresponding isonoise term in the isodiffusion representation. The latter definition of the isolaplacian differs from the original one introduced by
.

13.2. Diffusions and the Heisenberg Representation

Up to now we have set our theory in terms of the Schroedinger representation, since the original setting for this theory has to do with scale transformations as introduced by Einstein in his last work [14] while it was recognized previously by London that the wave function was related to the Weyl scale transformations, and these scale fields turned to be in the non-relativistic case, nothing else than the wave function of Schroedinger equation, both in the linear and the non-linear cases. Historically the operator theory of QM was introduced before the Schroedinger equation, who later proved the equivalence of the two. The ensuing dispute and rejection by Heisenberg of Schroedinger's equation is a dramatic chapter of the history of QM. It turns out to be the case that we can connect the Brownian motion approach to QM and the operator formalism due to Heisenberg and Jordan, and its isotopic lift presented in [72, 73]. We shall present this issue in the following.

Let us define the position operator as usual and the momentum operator by

$$q^k = x^k, \quad p_{\mathcal{D}k} = \sigma \times \frac{\partial}{\partial x^k}, \tag{167}$$

which we call the diffusion quantization rule (the subscript $\mathcal{D}$ denotes diffusion) since we have a representation different to the usual quantization rule

$$p_k = -i \times \frac{\partial}{\partial x^k}, \tag{168}$$

with $\sigma = (\sigma_a^\alpha)$ the diffusion tensor verifying $(\sigma \times \sigma^\dagger)^{\alpha\beta} = g^{\alpha\beta}$ and substitute into the Hamiltonian function

$$H(p,q) = \frac{1}{2}\sum_{k=1}^{d}(p_k)^2 + \mathbf{v}(q), \tag{169}$$

this yields the formal generator of a diffusion semigroup in $C^2(R^d)$ or $L^2(R^d)$ which in our previous notation is written as $H(g,0)+v$. Thus, an operator algebra on $C^2(R^n)$ or $L^2(R^n)$ together with the postulate of the commutation relation (instead of the usual commutator relation of quantum mechanics $[p,q] = -i \times I$)

$$[p_{\mathcal{D}}, q] = p_{\mathcal{D}} \times q - q \times p_{\mathcal{D}} = \sigma \times I \tag{170}$$

this yields the diffusion equation

$$\frac{\partial \phi}{\partial t} \times \phi + \frac{1}{2}\sum_{k=1}^{d}\left(\sigma\frac{\partial}{\partial x^\alpha}\right)^2 \times \phi + \mathbf{v} \times \phi = 0, \tag{171}$$

which coincides with the diffusion eq. (54) provided that $c = \mathbf{v}$. Thus, in this approach, the operator formalism and the "quantization postulates", allow to deduce the diffusion equation. If we start from either the diffusion process or the RCW geometry, without any quantization conditions we already have the equations of motion of the quantum system which are non other than the original diffusion equations, or equivalently, the Schroedinger equations. We stress the fact that these arguments are valid for both cases relative to the choice of the potential function V, i.e. if it depends nonlinearly on the wave function ψ, or acts linearly by multiplication on it. Further below, we shall use this modification of the Heisenberg representation of QM by the previous Heisenberg type representation for diffusion processes, to give an account of the diffusion processes that are associated with HM. This treatment differs from the original (inconsistent with respect to HM, as it turned to be proved in the later findings by Santilli) approach to isoquantization.

Let us frame now isoquantization in terms of diffusion processes. Define isomomentum, $\hat{p}_{\mathcal{D}}$, by

$$\hat{p}_{\mathcal{D}k} = \hat{\sigma}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^k}, \quad \text{with} \quad \hat{\sigma} = \sigma \times \hat{I}, \tag{172}$$

so that the kinetic term of the iso-Hamiltonian is

$$\begin{aligned}
\hat{p}_{\mathcal{D}}\hat{\times}\hat{p}^{\dagger}_{\mathcal{D}} &= \hat{\sigma}\hat{\times}\hat{\sigma}^{\dagger}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}} \\
&= \hat{g}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}} = \hat{\triangle}_{\hat{g}}
\end{aligned} \tag{173}$$

We finally check the consistency of the construction by proving that it can be achieved via the non-unitary transformation

$$\begin{aligned}
p_{\mathcal{D}_j} \mapsto U \times p_{\mathcal{D}_j} \times U^{\dagger} &= U \times \sigma \times \frac{\partial}{\partial x^j} \times U^{\dagger} \\
&= \sigma \times \hat{I} \times \hat{T} \times \hat{I} \times \frac{\partial}{\partial x^j} = \hat{\sigma}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^j} = \hat{p}_{\mathcal{D}j}.
\end{aligned} \tag{174}$$

Note that we have achieved this isoquantization in terms of the following transformations: Firstly, we carried out the transformation

$$p = -i \times \frac{\partial}{\partial x} \to p_{\mathcal{D}} := \sigma \times \frac{\partial}{\partial x}, \tag{175}$$

to further produce its isotopic lift

$$\hat{p}_{\mathcal{D}} = \hat{\sigma}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}}. \tag{176}$$

Whenever the original diffusion tensor σ is the identity I, from eq. (9) follows that the original metric g is Euclidean, we reach compatibility of the diffusion quantization with the Santilli-iso-Heisenberg representation given by taking the non-unitary transformation on the canonical commutation relations, which are given by

$$[\hat{q}^i \hat{,} \hat{p}_j] = \hat{i}\hat{\times}\hat{\delta}^i_j = i \times \delta^i_j \times \hat{I}, \tag{177}$$

together with

$$[\hat{r}^i \hat{,} \hat{r}^j] = [\hat{p}_i \hat{,} \hat{p}_j] = 0, \tag{178}$$

with the Santilli-iso-quantization rule [84]

$$\hat{p}_j = -\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^j}. \tag{179}$$

Thus, from the quantization by the diffusion representation we retrieve the Santilli-iso-Heisenberg representation, with the difference that the diffusion noise tensor in the above construction need not be restricted to the identity.

Finally, we consider the iso-Hamiltonian operator

$$\hat{H} = \frac{\hat{1}}{\hat{2}\hat{\times}\hat{m}}\hat{\times}\hat{p}^{\hat{2}} + \hat{V}_0(\hat{t},\hat{x}) + \hat{V}_k(\hat{t},\hat{v})\hat{\times}\hat{v}^k, \tag{180}$$

where $\hat{p}$ may be taken to be given either by the Santilli iso-quantization rule

$$\hat{p}_j\hat{\times}|\hat{\psi}> = -\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^j}\hat{\times}|\hat{\psi}>, \tag{181}$$

or by the diffusion representation $\hat{p}_{\mathcal{D}}$. $\hat{V}_0(\hat{t},\hat{x})$ and $\hat{V}_k(\hat{t},\hat{v})$ are potential isofunctions, the latter dependent on the isovelocities. Then the iso-Schroedinger equation (or Schroedinger-Santilli isoequation) [84] is

$$\begin{aligned} \hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{t}}|\hat{\psi}> &= \hat{H}\hat{\times}|\psi> \\ &= \hat{H}(\hat{t},\hat{x},\hat{p})\times\hat{T}(\hat{t},\hat{x},\hat{\psi},\hat{\partial}\hat{\psi},\ldots)\times|\hat{\psi}>, \end{aligned} \tag{182}$$

where the wave isofunction $\hat{\psi}$ is an element in $(\hat{\mathcal{H}}, <|\hat{\times}|>, \hat{C}(\hat{c},\hat{+},\hat{\times}))$ satisfies

$$\hat{I}\hat{\times}|\hat{\psi}> = |\hat{\psi}>. \tag{183}$$

13.3. Hadronic Mechanics and Diffusion Processes

Finally, the components of drift isovector field, projected on $\hat{M}(\hat{x},\hat{g},\hat{R})$ in the isotopic lift of eq. (115) is given by eq. (155) with $\hat{f} = \hat{\ln}\hat{\phi}$, where $\hat{\phi}(\hat{x}) = \hat{e}^{\hat{\mathcal{R}}(\hat{x})\hat{+}\hat{S}(\hat{x})}$ is the diffusion wave associated to the solution $\hat{\psi}(\hat{x}) = \hat{e}^{\hat{\mathcal{R}}(\hat{x})\hat{+}\hat{i}\hat{S}(\hat{x})}$ of the iso-Schroedinger equation, and its adjoint wave is $\hat{\phi}(x) = \hat{e}^{\hat{\mathcal{R}}(x)\hat{-}\hat{S}(x)}$. Hence, the drift isovector field has components

$$\hat{g}^{\alpha\beta}(\hat{x})\hat{\times}\frac{\hat{\partial}\hat{\ln}\hat{\phi}(\hat{x})}{\hat{\partial}\hat{x}^{\alpha}} = \hat{g}^{\alpha\beta}(\hat{x})\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}(\hat{\mathcal{R}}_{\hat{t}}\hat{+}\mathcal{S}_{\hat{t}})(\hat{x}), \tag{184}$$

Finally, we shall write the isotopic lift of the stochastic differential equation for the iso-Schroedinger eq. (107). Applying the non-unitary transformation to eq. (63), we obtain the iso-equation on $\hat{M}(\hat{x},\hat{G},\hat{R})$ for $\hat{X}_{\hat{t}}$ given by

$$d\hat{X}^i_{\hat{t}} = ((\hat{g}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}(\hat{\mathcal{R}}_{\hat{t}}\hat{+}\mathcal{S}_{\hat{t}}))(\hat{X}_{\hat{t}})\hat{\times}\hat{d}\hat{t}\hat{+}\hat{\sigma}^i_j(\hat{X}_{\hat{t}})\hat{\times}d\hat{W}^j_{\hat{t}}, \tag{185}$$

with $d\hat{W}_{\hat{t}} = \hat{W}(\hat{t}\hat{+}\hat{d}\hat{t})\hat{-}\hat{W}(\hat{t})$ the increment of a iso- Wiener process $\hat{W}_{\hat{t}} = (\hat{W}^1_{\hat{t}},\ldots,\hat{W}^m_{\hat{t}})$ with isoaverage equal to $\hat{0}$ and isocovariance given by $\hat{\delta}^i_j\hat{\times}\hat{t}$; i.e.,

$$\hat{1}\hat{/}(\hat{4}\hat{\times}\hat{\pi}\hat{\times}\hat{t})^{\hat{m}\hat{/}\hat{2}}\hat{\int}\hat{w}_i\hat{\times}\hat{e}^{-\hat{w}^{\hat{2}}\hat{/}\hat{4}\times\hat{t}^{\hat{2}}}\hat{\times}\hat{d}\hat{w} = \hat{0}, \quad \forall i = 1,\ldots,m \tag{186}$$

and

$$\hat{1}\hat{/}(\hat{4}\hat{\times}\hat{\pi}\hat{\times}\hat{t})^{\hat{m}\hat{/}\hat{2}}\hat{\int}\hat{w}_i\hat{\times}\hat{w}_j\hat{\times}\hat{e}^{-\hat{w}^{\hat{2}}\hat{/}\hat{4}\hat{\times}\hat{t}^{\hat{2}}}\hat{\times}\hat{d}\hat{w} = \hat{\delta}^i_j\hat{\times}\hat{t}, \quad \forall i,j = 1,\dots,m \tag{187}$$

and $\hat{\int}$ denotes the isotopic integral defined by $\hat{\int}\hat{d}\hat{x} = (\int \hat{T}\times\hat{I}\times dx)\times\hat{I} = (\int dx)\times\hat{I} = \hat{x}$. Thus, formally at least, we have

$$\hat{X}_{\hat{t}} = \hat{X}_{\hat{0}}\hat{+}\hat{\int}_{\hat{0}}^{\hat{t}}\Big(\hat{g}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}(\hat{\mathcal{R}}_{\hat{s}}\hat{+}\mathcal{S}_{\hat{s}})\Big)(\hat{X}_{\hat{s}})\hat{\times}\hat{d}\hat{s}\hat{+}\hat{\int}_{\hat{0}}^{\hat{t}}\hat{\sigma}^i_j(\hat{X}_{\hat{s}})\hat{\times}d\hat{W}^j_{\hat{s}}. \tag{188}$$

The integral in the first term of eq. (189) is an isotopic lift of the usual Riemann-Lebesgue integral [31], while the second one is the isotopic lift of a stochastic Itô integral; we shall not present here in detail the definition of this last term, which follows from the notions of convergence in the isofunctional analysis (see [84]), and the usual definition of Itô stochastic integrals [30, 46], nor the presentation of analytical conditions for their convergence which follows in principle from the isotopic lift of the usual conditions.

14. Statistical Thermodynamics: Preliminaries

Up to now we have established a theory in which Brownian motions and RCW geometries appeared to be two faces of the same phenomenae: fluctuations and RCW geometries are fused. We shall go one step further to see how they are further related in the study of non-equilibrium systems in terms of statistical thermodynamics, as envisaged by Stratonovich [90]. The studies of fluctuation-dissipation relations are at the core of this approach which was somewhat initiated himself by Einstein with his studies in Brownian motion. We are particularly interested in the non-linear Boltzmann H-theorem, and its relation with torsion-drift which we shall elaborate following Stratonovich [90] and the presentation by Rapoport [66].

Suppose we have random variables $\xi_1,\dots,\xi_n$, (n is an arbitrary positive integer) with joint probability density $p(\xi_1,\dots,\xi_n)$. The mean value

$$<\xi_1,\dots,\xi_n> = \int \xi_1\dots\xi_n p(\xi_1,\dots,\xi_n)d\xi_1\dots d\xi_n, \tag{189}$$

is called the moment. The characteristic function, $\Xi(iu_1,\dots,iu_n)$, is defined by

$$\begin{aligned}\Xi(iu_1,\dots,iu_n) &= <\exp\Big(\sum_{\alpha=1}^{n} iu_\alpha\xi_\alpha\Big)> \\ &= \int\exp\Big(\sum_{\alpha=1}^{n} iu_\alpha\xi_\alpha\Big)p(\xi_1,\dots,\xi_n)d\xi_1\dots d\xi_n.\end{aligned} \tag{190}$$

Then, it follows that the moments can be expressed in terms of the characteristic function:

$$<\xi_{\alpha_1},\dots,\xi_{\alpha_m}> = i^{-m}\Big[\frac{\partial\Xi(iu_1,\dots,iu_m)}{\partial u_{\alpha_1}\dots\partial u_{\alpha_m}}\Big]\Big|_{u=0}, \quad 1\leq m, \tag{191}$$

where $u = 0$ means $u_1 = \dots = u_m = 0$.

If the characteristic function is analytic at the points where $u = 0$, the Taylor formula is valid

$$\begin{aligned}\Xi(iu_1,\ldots,iu_n) &= 1+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\alpha_1\ldots\alpha_m}\Big[\frac{\partial\Xi(iu_1,\ldots,iu_m)}{\partial u_{\alpha_1}\ldots\partial u_{\alpha_m}}\Big]\Big|_{u=0}u_{\alpha_1}\ldots u_{\alpha_m}\\ &= 1+\sum_{m=1}^{\infty}\frac{i^m}{m!}\sum_{\alpha_1\ldots\alpha_m}<\xi_{\alpha_1}\ldots\xi_{\alpha_m}>u_{\alpha_1}\ldots u_{\alpha_m}, \qquad (192)\end{aligned}$$

since from eq. (190) follows that $\Xi(0) = 1$. Formula (192) can only be used when all the moments are finite and the expansion in (192) converges. Thus , the characteristic function can be represented in terms of the moments. We further introduce the *correlator* defined by

$$<\xi_{\alpha_1},\ldots,\xi_{\alpha_m}>= i^{-m}\Big[\frac{\partial^m \ln\Xi(iu_1,\ldots,iu_m)}{\partial u_{\alpha_1}\ldots\partial u_{\alpha_m}}\Big]\Big|_{u=0}. \qquad (193)$$

We note that the correlator of the random variables $\xi_{\alpha_1},\ldots,\xi_{\alpha_m}$ is 0 if these variables can be divided at least into two groups such that one group is statistically independent from the other group. If, in fact, say $\xi_{\alpha_1},\ldots,\xi_{\alpha_k}$ are statistically independent of $\xi_{\alpha_{k+1}},\ldots,\xi_{\alpha_m}$, then the probability density is factorizable as

$$p(\xi_{\alpha_1},\ldots,\xi_{\alpha_m}) = p_1(\xi_{\alpha_1},\ldots,\xi_{\alpha_k})p_2(\xi_{\alpha_{k+1}},\ldots,\xi_{\alpha_m}), \qquad (194)$$

and the characteristic function factorizes as

$$\Xi(iu_{\alpha_1},\ldots,iu_{\alpha_m}) = \Xi_1(iu_{\alpha_1},\ldots,iu_{\alpha_k})\Xi_2(iu_{\alpha_{k+1}},\ldots,iu_{\alpha_m}). \qquad (195)$$

Then, writing $v_\alpha = iu_\alpha$ for any α, we have

$$\ln\Xi(v_{\alpha_1},\ldots,v_{\alpha_m}) = \ln\Xi_1(v_{\alpha_1},\ldots,v_{\alpha_k})+\ln\Xi_2(v_{\alpha_{k+1}},\ldots,v_{\alpha_m}), \qquad (196)$$

and then from eqs. (193, 196) follows that

$$<\xi_{\alpha_1},\ldots,\xi_{\alpha_m}>= \frac{\partial^m \ln\Xi_1(v_{\alpha_1},\ldots,v_{\alpha_k})}{\partial v_{\alpha_1}\ldots\partial v_{\alpha_k}\partial v_{\alpha_{k+1}}\ldots\partial v_{\alpha_m}}+\frac{\partial^m \ln\Xi_2(v_{\alpha_{k+1}},\ldots,v_{\alpha_m})}{\partial v_{\alpha_1}\ldots\partial v_{\alpha_k}\partial v_{\alpha_{k+1}}\ldots\partial v_{\alpha_m}} \qquad (197)$$

is equal to 0. We have from eq. (193), if the expression (192) converges,

$$\Xi(v_1,\ldots,v_n) = \exp\Big(1+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\alpha_1,\ldots,\alpha_m=1}^{m}<\xi_{\alpha_1},\ldots,\xi_{\alpha_m}>v_{\alpha_1}\ldots v_{\alpha_m}\Big). \qquad (198)$$

Therefore, from eq. (192) follows that

$$\Xi(iu_1,\ldots,iu_m) = 1+\sum_{m=1}^{\infty}\frac{i^m}{m!}\sum_{\alpha_1\ldots\alpha_m}<\xi_{\alpha_1}\ldots\xi_{\alpha_m}>u_{\alpha_1}\ldots u_{\alpha_m}, \qquad (199)$$

which is still equal to

$$\exp\Big(\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{\beta_1,\ldots,\beta_k=1}^{n}<\xi_{\beta_1},\ldots,\xi_{\beta_k}>v_{\beta_1}\ldots v_{\beta_k}\Big). \qquad (200)$$

This is called the generating equation, since multiple partial differentiation with respect to the components of $v = (v_1, \ldots, v_n)$ and further setting $v = 0$, enables us to express the moments in terms of the correlators. Thus, for example, double differentiation yields

$$< \xi_{\alpha_1}\xi_{\alpha_2} >=< \xi_{\alpha_1}, \xi_{\alpha_2} > + < \xi_{\alpha_1} >< \xi_{\alpha_2} > . \tag{201}$$

If we calculate the third derivative and we further set v equal to 0, we obtain

$$\begin{aligned} < \xi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_3} > &= < \xi_{\alpha_1}, \xi_{\alpha_2}\xi_{\alpha_3} > + < \xi_{\alpha_1}, \xi_{\alpha_3} >< \xi_{\alpha_2} > + < \xi_{\alpha_2}, \xi_{\alpha_3} >< \xi_{\alpha_1} > \\ &+ < \xi_{\alpha_1}, \xi_{\alpha_2} >< \xi_{\alpha_3} > + < \xi_{\alpha_1} >< \xi_{\alpha_2} >< \xi_{\alpha_3} > \end{aligned} \tag{202}$$

which we can write in the shorter form

$$\begin{aligned} < \xi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_3} > &= < \xi_{\alpha_1}, \xi_{\alpha_2}\xi_{\alpha_3} > +(3) < \xi_{\alpha_1}, \xi_{\alpha_3} >< \xi_{\alpha_2} > \\ &+ < \xi_{\alpha_1} >< \xi_{\alpha_2} >< \xi_{\alpha_3} > \end{aligned} \tag{203}$$

where the term with the coefficient (3) indicates the number of terms of the same type, differing only in the order of the subscripts. These formulas have the following characteristic feature: in the right hand size appear all possible terms (with Pascal triangle coefficients) that are different despite the symmetry of correlators and products that correspond to various groupings of the elements $\xi_{\alpha_1}, \ldots, \xi_{\alpha_n}$.

15. Markov Processes

Let $x(\tau) = (x_1(\tau), \ldots, x_r(\tau)))$ be an r-component random process. It is described by the set of many-time-τ probability densities

$$p(x(\tau)), p(x(\tau_1), x(\tau_2)), \ldots, p(x(\tau_1), x(\tau_2), \ldots, x(\tau_n)), \ldots . \tag{204}$$

Next we introduce the conditional probability density

$$p(\eta|\xi) = \frac{p(\eta, \xi)}{p(\xi)}. \tag{205}$$

Thus, putting $\eta = x(\tau_2), \xi = x(\tau_1)$, we have

$$p(x(\tau_1), x(\tau_2)) = p(x(\tau_2)|x(\tau_1))p(x(\tau_1)). \tag{206}$$

If instead we put $\xi = x(\tau_3), \eta = (x(\tau_2), x(\tau_1))$, we have from eq. (205)

$$p(x(\tau_1), x(\tau_2), x(\tau_3)) = p(x(\tau_3)|x(\tau_1), x(\tau_2)).p(x(\tau_2)|x(\tau_1)).p(x(\tau_1)), \tag{207}$$

or

$$p(x(\tau_1), x(\tau_2), x(\tau_3)) = p(x(\tau_1), x(\tau_2))p(x(\tau_3)|x(\tau_1)).p(x(\tau_2)). \tag{208}$$

Generally, we have

$$\begin{aligned} p(x(\tau_1), \ldots, x(\tau_n)) &= p(x(\tau_1))p(x(\tau_2)|x(\tau_1))p(x(\tau_3)|x(\tau_1), x(\tau_2)) \\ &\quad \ldots p(x(\tau_n)|x(\tau_1), \ldots, x(\tau_{n-1})). \end{aligned} \tag{209}$$

Assume that $\tau_1 < \tau_2 < \ldots < \tau_n$. A r-component stochastic process $x(\tau)$ is called Markovian, if for any $k \in N$,

$$p(x(\tau_k)|x(\tau_1),\ldots,x(\tau_{k-1})) = p(x(\tau_k|x(\tau_{k-1})), \tag{210}$$

is valid. This condition is usually framed as saying that Markovian process are memoryless, since the conditional probabilities depend on the previous predecessor but not on the other predecessors. In this case, for $\tau_1 < \tau_2 < \ldots < \tau_{n-1} < \tau_n$,

$$p(x(\tau_1),\ldots,x(\tau_n)) = p(x(\tau_n)|x(\tau_{n-1}))\ldots p(x(\tau_3)|x(\tau_2))p(x(\tau_2)|x(\tau_1)))p(x(\tau_1)). \tag{211}$$

Thus, we conclude that for a Markovian process, the many-time probability densities are determined by the one-time probability density $p(x(\tau))$ and the two-time conditional probability density $p(x(\tau)|x(\tau')), \tau' \leq \tau$, i.e. the transition probability (density).

Now if we integrate the three-time probability density $p(x(\tau_1),x(\tau_2),x(\tau_3))$ with respect to $x(\tau_2)$ along the reals, we obtain the two-time probability density (here, $\tau_1 < \tau_2 < \tau_3$),

$$\int p(x(\tau_1),|x(\tau_2))p(x(\tau_2)|x(\tau_1))dx(\tau_2) = p(x(\tau_3)|x(\tau_1)). \tag{212}$$

Now, representing both probability densities in the form of eq. (211) and dropping $p(x(\tau_1)$ we obtain the equation

$$\int\int p(x(\tau_3)|x(\tau_2))p(x(\tau_2)|x(\tau_1))dx(\tau_2) = p(x(\tau_3|x(\tau_1)),(\tau_1 < \tau_2 < \tau_3). \tag{213}$$

This is the Chapman-Kolgomorov-Smoluchowski (CKS) equation which is satisfied by the transition probability densities. Denoting $x(\tau_1) = x'', x(\tau_2) = x'$, we can write the transition probability density $p(x(\tau_2)|x(\tau_1))$ as $p_{\tau_2\tau_1}(x',x'')$. Then, the CKS equation yields for $x = x(\tau_3), x' = x(\tau_2), x'' = x(\tau_1)$, the equation

$$\int p_{\tau_3\tau_2}(x|x')p_{\tau_2\tau_1}(x'|x'')dx' = p_{\tau_3\tau_1}(x|x''). \tag{214}$$

Let us introduce the conditional probability characteristic function of increments $\triangle x = x - x' = x(\tau_3) - x(\tau_2)$, as the Fourier transform of $p_{\tau_3\tau_2}(x|x')$:

$$\Xi_{\tau_3\tau_2}(iu|x') = \int \exp(-iu(x-x'))p_{\tau_3\tau_2}(x|x')dx, \tag{215}$$

where $u\triangle x = \sum_{\alpha=1}^{r} u_\alpha \triangle x_\alpha$. Using the inverse Fourier transform we can express the transition probability density $p_{\tau_3\tau_2}(x|x')$ in terms of the characteristic function

$$p_{\tau_3\tau_2}(x|x') = (2\pi)^{-r}\int \exp[-iu(x-x')]\Xi_{\tau_3\tau_2}(iu|x'')du. \tag{216}$$

Inserting this into the CKS equation we get

$$(2\pi)^{-r}\int \exp[-iu(x-x')]\Xi_{\tau_3\tau_2}(iu|x')p_{\tau_2\tau_1}(x'|x'')dudx' = p_{\tau_3\tau_1}(x|x''). \tag{217}$$

This last equation is easily seen to be equal to

$$(2\pi)^{-r}\int \exp[-iu(x-x')]\Xi_{\tau_3\tau_2}\Big(-\frac{\partial}{\partial x'}|x'\Big)p_{\tau_2\tau_1}(x'|x'')dudx' = p_{\tau_3\tau_1}(x|x''). \tag{218}$$

We integrate this last equation with respect to u by applying the integral representation of the delta function

$$(2\pi)^{-r}\int_{-\infty}^{+\infty}\exp(iuz)du = \delta(z) \equiv \delta(z_1)\ldots\delta(z_r). \tag{219}$$

Then, we have

$$\int \Xi_{\tau_3\tau_2}(-\frac{\partial}{\partial x'}|x')\delta(x-x')p_{\tau_2\tau_1}(x'|x'')dudx' = p_{\tau_3\tau_1}(x|x''). \tag{220}$$

This integration trivially yields the CKS characteristic function equation

$$N_{\partial,x}\Xi_{\tau_3\tau_2}\Big(-\frac{\partial}{\partial x'}|x\Big)p_{\tau_3\tau_1}(x|x'') = p_{\tau_3\tau_1}(x|x''). \tag{221}$$

The last inserted symbol, $N_{\partial,x}$, indicates that the differentiation is done after multiplication by functions of x of the transition density has been carried out on the right. CKS and eq. (221) are equivalent. Using now the Taylor expansion of the characteristic function given by eqs. (199, 200), we obtain the following form of the CKS equation

$$\begin{aligned} p_{\tau_3\tau_1}(x|x'') &= \sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{\alpha_1,\ldots,\alpha_m=1}^{r}\frac{\partial^m}{\partial x^{\alpha_1}\ldots\partial x^{\alpha_m}}[< \triangle x_{\alpha_1}\ldots\triangle x_{\alpha_m} >_x p_{\tau_2\tau_1}(x|x'')] \\ &+ p_{\tau_2\tau_1}(x|x''). \end{aligned} \tag{222}$$

Here the subscript x shows that the moments are in fact conditional, i.e. they are taken at fixed valued of $x = x(\tau_2)$.

Let $\tau = \tau_3 - \tau_2 > 0$. We write the CKS characteristic function equation in the form

$$\tau^{-1}[p_{\tau_3\tau_1}(x,x'') - p_{\tau_2\tau_1}(x|x'')] = \tau^{-1}[N_{\partial,x}\Xi_{\tau_3\tau_2}(-\frac{\partial}{\partial x}|x) - 1]p_{\tau_2\tau_1}(x|x''). \tag{223}$$

If we let τ tend to 0 in this equation, provided the limit

$$\Phi_{\tau_2}(v,x) = \lim_{\tau\to 0}\frac{\Xi_{\tau_2+\tau}(v|x) - 1}{\tau}, \tag{224}$$

exists, we obtain the equation

$$\frac{\partial p_{\tau_2\tau_1}}{\partial\tau_2}(x|x'') = N_{\partial,x}\Phi_{\tau_2}(-\frac{\partial}{\partial x},x)p_{\tau_2\tau_1}(x|x''). \tag{225}$$

This is the *master* equation. The initial condition for the master equation is

$$p_{\tau_2\tau_1}(x|x'') = \delta(x-x''), \text{for } \tau_2 = \tau_1. \tag{226}$$

Therefore, if the function $\Phi_\tau(v,x)$ is known, one can find the transition densities as the fundamental solution of the master equation. This function is called the *Markov generator function*.

Thus we see that the operator $L_\tau = N_{\partial,x}\Phi_\tau(-\frac{\partial}{\partial x},x)$ defines the statistics of the Markov process. This operator is called the Markov operator, or still, the *differential operator* or the *infitesimal generator of the Markov process*. This is the fundamental structure for the construction of the Markov process. Yet, we must stress as presented here, has the most serious problem of being non-covariant, thus the physical interpretations of the terms of this operator is unclear unless we promote this operator to a covariant form defined by geometrical constructions. This will turn to be the case in the restricted form of the master equation defined up to second-order derivatives, and then the differential operator turns to be the Laplacian operator of a geometry with trace-torsion, and viceversa, this geometry is determined by the one-time transition densities. Let us analyze this further.

Multiplying the master equation by $p_{\tau_1}(x) = p(x(\tau_1))$, and further integrating with respect to $x'' = x(\tau_1)$, we find that the one-time probability density

$$\frac{\partial p_\tau(x)}{\partial \tau} = N_{\partial,x}\Phi_\tau(-\frac{\partial}{\partial x},x)p_\tau(x). \tag{227}$$

The previous process of taking the limit with $\tau \to 0$ can also be carried out for $p_\tau(x)$. Then, the master equation takes the form

$$\frac{\partial p_\tau(x)}{\partial \tau} = \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \sum_{\alpha_1\ldots\alpha_m=1}^{r} \frac{\partial^m}{\partial x_{\alpha_1}\ldots\partial x_{\alpha_m}}[\Lambda^{\alpha_1\ldots\alpha_m}(x)p_\tau(x)], \tag{228}$$

where

$$\Lambda^{\alpha_1\ldots\alpha_m}(x) = \lim_{\tau\to 0} < \triangle x_{\alpha_1}\ldots\triangle x_{\alpha_m} >_x . \tag{229}$$

The Markov generator function $\Phi(v,x)$ and the coefficients $\Lambda^{\cdots}$ defined in (229), are related by

$$\Phi(v,x) = \sum_{m=1}^{\infty}\sum_{\alpha_1\ldots\alpha_m} \Lambda^{\alpha_1\ldots\alpha_m}(x)v_{\alpha_1}\ldots v_{\alpha_m}, \tag{230}$$

if this series converges. In this case the identity is called the Kramers-Moyal equation. The functions $\Lambda^{\cdots}$ determine the statistics of the process, and are called the *coefficients of the master equation*, or still, the *coefficient functions*.

In this article we shall focuse our attention in the case that only the coefficients $\Lambda^\alpha(x), \Lambda^{\alpha\beta}(x)$ are the only ones not vanishing identically. This is the case of the *Fokker-Planck operator* and ultimately, the case of a geometry defined by a Riemann-Cartan-Weyl geometry, with a metric and a trace-torsion as defined above as the trace of the torsion tensor. Thus, the Fokker-Planck equation is

$$\frac{\partial p_\tau(x)}{\partial \tau} = -\frac{\partial}{\partial x^\alpha}[\Lambda^\alpha(x)p_\tau(x) + \frac{1}{2}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}[\Lambda^{\alpha\beta}(x)p_\tau(x)]. \tag{231}$$

We confront this non-covariant form of the Fokker-Planck equation with the form

$$\frac{\partial p_\tau(x)}{\partial \tau} = -\mathrm{div_g}[b(x)p_\tau(x)] + \frac{1}{2}div_g[\mathrm{grad}_g p_\tau(x)] \tag{232}$$

where $b(x) = b^\alpha(x)\frac{\partial}{\partial x^\alpha}$ is the drift vectorfield which is related to the trace-torsion one-form $Q = Q_\alpha dx^\alpha$ by

$$b^\alpha = g^{\alpha\beta}Q_\beta(x). \tag{233}$$

Furthermore, div_g is the metric divergence operator defined by

$$div_g(X) = \frac{1}{\det(g)^{\frac{1}{2}}} \frac{\partial \det(g)^{\frac{1}{2}} X^{\beta}}{\partial x^{\beta}}, \tag{234}$$

for any vectorfield $X = X^{\beta}\frac{\partial}{\partial x^{\beta}}$, and for a function f (say of class C^2) $\mathrm{grad}_g f = g^{\alpha\beta}\frac{\partial f}{\partial x^{\beta}}\frac{\partial}{\partial x^{\alpha}}$, so that the second-order term

$$\frac{1}{2} div_g[\mathrm{grad}_g p_{\tau}(x)] = \frac{1}{2}\triangle_g p_{\tau}(x), \tag{235}$$

is the action of the invariant second-order geometrical Laplace-Beltrami operator defined by the metric g, while the first-order term is also geometrical and invariant defined by the first divergence term in which the drift-torsion $b(x)$ is present. For simplicity, instead of rewriting the previous deductions using the invariant volume element $\det(g)^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n$ defined by the metric, we shall rescale the metric $g^{\alpha\beta}$, if necessary (this rescaling is tantamount to rescale the diffusion constant, thus of no fundamental significance), with a factor $\det(g)^{\frac{1}{2}}$, so that we can set this latter term equal to 1 and thus

$$\Lambda^{\alpha\beta} = g^{\alpha\beta}, \forall \alpha, \beta, \tag{236}$$

and we also have

$$\Lambda^{\alpha} = b^{\alpha} \equiv g^{\alpha\beta} Q_{\beta}, \forall \alpha. \tag{237}$$

Therefore, we can conclude relating the master equation to a RCW laplacian operator. Indeed, with g, Q related by eqs. (233), takes the form

$$\frac{\partial p_{\tau}(x)}{\partial \tau} = -\mathrm{div_g}[b(x) p_{\tau}(x)] + \frac{1}{2}\triangle_g p_{\tau}(x). \tag{238}$$

In this case the Fokker-Planck operator is defined by a geometric structure; in fact, it is the adjoint

$$H_0(g,Q)^{\dagger} p_{\tau}(x) = \frac{1}{2}\triangle_g p_{\tau}(x) - \mathrm{div_g}\left[b^{\alpha}(x)\frac{\partial p_{\tau}(x)}{\partial x^{\alpha}}\right], \tag{239}$$

of the Laplacian operator $H_0(g,Q)$ defined by a Riemann-Cartan-Weyl connection defined by a metric g and a trace-torsion one-form Q. In this case the statistics of the system is completely determined by the geometry, and viceversa, the statistics of the system define this geometry completely. Thus we have turned eq. (231) into a geometric-derived equation with right hand side given in eq. (239).

16. The Canonical Representation of the Markov Function

We would like to discuss further the diffusion processes and their geometry in the more general framework of Markov processes. We start by introducing the notion of infinite divisible probability densities.

16.1. Infinite Divisible Probability Densities

We start with the case of random variables, i.e. for unidimensional processes to give the decomposition for vector-valued processes. Let ξ be a random variable and $p(\xi)$ its probability density. Then, ξ is called infinite-divisible, if there exists a sequence of random variables $\eta_j, j \in \{1,\ldots,n\}$, with n arbitrary large, all having the same probability density $p_n(\eta)$, such that

$$\xi = \eta_1 + \ldots + \eta_n. \tag{240}$$

Let

$$\Xi(iu) = \int \exp(iu\xi)p(\xi)d\xi, \tag{241}$$

and

$$\Xi_n(iu) = \int \exp(iu\xi)p_n(\xi)d\xi, \tag{242}$$

the characteristic functions. Recalling the basic property that the characteristic function of statistically independent random variables equals the product of the characteristic functions of each variable, and further, since the variables have the same probability density and thus the same characteristic function Ξ_n, we conclude that the characteristic value of the sum ξ is equal to the power n of the characteristic function $\Xi_n(iu)$, i.e.

$$\Xi(iu) = [\Xi_n(iu)]^n. \tag{243}$$

This means that the condition of infinite divisibility of p_ξ reduces to the fact that $[\Xi(iu)]^{\frac{1}{n}}$ is the characteristic function of a random variable, or equivalently, the inverse of the Fourier transform of $[\Xi(iu)]^{\frac{1}{n}}$ is a probability density, i.e.

$$p_n(\eta) = (2\pi)^{-1}\int \exp(-iu\eta)[\Xi(iu)]^{\frac{1}{n}}du. \tag{244}$$

Thus, p_η is non-negative and normalized. Indeed, we have

$$\begin{aligned}\int p_n(\eta)d\eta &= \int (2\pi)^{-1}\int d\eta \exp(-iu\eta)[\Xi(iu)]^{\frac{1}{n}} \\ &= \int \delta(u)[\Xi(iu)]^{\frac{1}{n}}du = [\Xi(0)]^{\frac{1}{n}} = 1.\end{aligned} \tag{245}$$

Therefore, we have proved tht p_η is normalized. Yet, it is known [24], that a a probability density, $p(\xi)$ satisfying the condition,

$$\int \xi^2 p(\xi)d\xi, \tag{246}$$

is infinitely divisible if and only if the integral representation of its characteristic function (the so called canonical fundamental representaton)

$$\ln \Xi(iu) = -i\gamma u + \int [\exp(iuz) - 1 - iuz]z^{-2}H(z)dz, \tag{247}$$

is valid. Here $H(z)$ is a non-negative integrable function, which can have singularities of the Dirac-delta type, and γ is a real constant.

The previous result for a unidimensional random variable can be generalized to a vector-valued random variable, $\xi = (\xi_1, \ldots, \xi_n)$. In this case η, u and z are vectors $(\eta_1, \ldots, \eta_n), u = (u_1, \ldots, u_n)$ and $z = (z_1, \ldots, z_n)$. So, $uz = u_\alpha z^{\alpha_1} = u_1 z_1 + \ldots + u_n z_n$ and $z^2 = (z_1)^2 + \ldots (z_n)^2$. (Here we have adopted again the Einstein convention of repeated indices meaning sum on all its possible values.)

Let us consider a stochastic process $y(\tau) = (y_1(\tau), \ldots, y_n(\tau))$ which we assume to be a stationary process. We shall say that this process has independent increments if for any instants such that $\tau_1 < \ldots < \tau_n$, the increments $y(\tau_2) - y(\tau_1), y(\tau_3) - y(\tau_2), \ldots, y(\tau_n) - y(\tau_{n-1})$, are statistically independent. It follows that for a stationary process with independent increments, the probability density $p(y(tau') - y(\tau))$ is infinitely divisible, for $\tau' > \tau$. Indeed, if we set $s = \frac{(\tau' - \tau)}{n}$, the increment can be represented as the sum of independent increments

$$y(\tau') - y(\tau) = [y(\tau + s) - y(\tau)] + [y(\tau + 2s) - y(\tau + s)] + \ldots + [y(\tau') - y(\tau' - s)], \quad (248)$$

for arbitrary positive n. Note that these increments have the same probability density due to the stationarity of the process. Therefore, if each summand has a finite second moment, the condition for the representation of above is valid, i.e. the characteristic function given by

$$\Xi_\tau(iu) = < \exp(iu[y(\tau + t) - y(\tau)] >= \quad (249)$$

can be represented as

$$\ln \Xi_\tau(iu) = i\gamma_\tau u + \int [\exp(iuz) - 1 - iuz] z^{-2} H_\tau(z) dz, \quad (250)$$

where $H_\tau(z)$ is a non-negative integrable function. This is the canonical representation mentioned before [24].

Now, the condition of independence of increments, implies that the characteristic function satisfies

$$\Xi_{\tau_1+\tau_2}(iu) = \Xi_{\tau_1}(iu) + \Xi_{\tau_2}(iu), \quad (251)$$

which implies that $\ln \Xi_\tau(iu)$ is proportional to τ, which we write as

$$\ln \Xi_\tau(iu) = \tau\phi(iu). \quad (252)$$

Comparison of eqs. (252) and (250) yields

$$\gamma_\tau = \gamma_1 \tau, H_\tau(z) = H_1(z)\tau, \quad (253)$$

where $H_1(z)$ is necessarily a non-negative integrable function.

The crucial fact is that a process with independent increments is a Markov process, yet its characteristic function loses its dependence on y' [24]. From eqs. (224) , after passing to the limit we get

$$\Phi(v) = \phi(v), \quad (254)$$

so that the master eq. (225) is

$$\frac{\partial p_\tau(y|y'')}{\partial \tau} = \phi(\frac{\partial}{\partial y}) p_\tau(y|y''). \quad (255)$$

Then, in consideration of eq. (253) we can write

$$\Phi(v) = \phi(v) = (\gamma^1)^\alpha v_\alpha + \int_{-\infty}^{+\infty} [\exp(vz) - 1 - vz] z^{-2} H_1(z) dz, Re(v) = 0. \tag{256}$$

Further comparison with the Kramers-Moyal expansion (eq. (230)), yields

$$(\gamma^1)^\alpha = \Lambda^\alpha, \forall \alpha, \tag{257}$$

and therefore,

$$\Phi(v) = \phi(v) = \Lambda^\alpha v_\alpha + \int_{-\infty}^{+\infty} [\exp(vz) - 1 - vz] z^{-2} H_1(z) dz, Re(v) = 0. \tag{258}$$

We can see clearly the drift term (see eq. (237)) in the first term of this canonical representation. This will be crucial to the sequel.

Of course, what matters here is the applicability of this representation to a general stationary Markov process. In this case, the probability densities of the increments $y(\tau') - y(\tau), \tau' > \tau$, are not strictly speaking infinitely divisible, unless we consider small increments $s = \tau' - \tau$, in which case the infinite divisibility of the increments is valid [90]. This is also a crucial fact.

16.2. The Kinetic Potential and Its Image

In this section, we shall consider a stationary Markov process, $B(\tau) = (B^1(\tau), \ldots, B^n(\tau))$, which we may think as thermodynamic equilibrium fluctuations. This process is characterized by the coefficient functions

$$\Lambda^{\alpha_1 \ldots \alpha_n} = \lim_{\tau \to 0} \frac{1}{\tau} < \triangle B^{\alpha_1} \ldots \triangle B^{\alpha_n} >_B, \tag{259}$$

where

$$\triangle B = B(\tau_1 + \tau) - B(\tau_1), B = B(\tau_1). \tag{260}$$

Henceforth we shall assume the previous limits to exist. Now we can deduce the following estimation formulas for small $\tau > 0$:

$$< \triangle B^{\alpha_1} \ldots \triangle B^{\alpha_n} >_B = \Lambda^{\alpha_1 \ldots \alpha_n}(B)\tau + o(\tau). \tag{261}$$

We shall use next the formulas

$$< \triangle B^{\alpha_1}, \triangle B^{\alpha_2} >_B = < \triangle B^{\alpha_1} \triangle B^{\alpha_2} >_B - < \triangle B^{\alpha_1} >_B < \triangle B^{\alpha_2} >_B, \tag{262}$$

and

$$\begin{aligned} < \triangle B^{\alpha_1}, \triangle B^{\alpha_2}, \triangle B^{\alpha_3} >_B &= -(3) < \triangle B^{\alpha_1}, \triangle B^{\alpha_2} >_B < \triangle B^{\alpha_3} >_B \\ &+ 2 < \triangle B^{\alpha_1} >_B < \triangle B^{\alpha_2} >_B < \triangle B^{\alpha_3} >_B \\ &+ < \triangle B^{\alpha_1} \triangle B^{\alpha_2} \triangle B^{\alpha_3} >_B, \end{aligned} \tag{263}$$

and higher-terms correlators, which are the inverse of the formulas in eqs. $(201-203)$. Inserting eq. (260) into them we get

$$< \triangle B^{\alpha_1},\ldots,\triangle B^{\alpha_n} >_B = \Lambda^{\alpha_1\ldots\alpha_n}(B)\tau + o(\tau). \tag{264}$$

Hence, the coefficients of the master equation can be also determined in terms of the correlators

$$\Lambda^{\alpha_1\ldots\alpha_n}(B) = \lim_{\tau\to 0}\frac{1}{\tau} < \triangle B^{\alpha_1},\ldots,\triangle B^{\alpha_n} >_B . \tag{265}$$

Now, if we assume that the equilibrium correlators have the polynomial form

$$< \triangle B^{\alpha_1},\ldots,\triangle B^{\alpha_n} >_B \approx k^{n-1}, n \geq 1 \tag{266}$$

where k is the Boltzmann constant, then we conclude that

$$\Lambda^{\alpha_1\ldots\alpha_n}(B) \approx k^{n-1}, n \geq 1. \tag{267}$$

Thus, the coefficients of the master equation and one-time equilibrium correlators have identical orders of magnitude. We introduce the non-equilibrium kinetic potential $V(y,B)$ by the formula

$$\Lambda^{\alpha_1\ldots\alpha_n}(B) = (kT)^{n-1}\Big[\frac{\partial^n V(y,B)}{\partial y^{\alpha_1}\ldots\partial y^{\alpha_n}}\Big]_{y=0}, \quad n \geq 1. \tag{268}$$

with T the temperature. Thus defined, the kinetic potential is a macroscopic quantity just like the free energy, i.e. itself and its derivatives with respect to both y and B are zero-th order in k, i.e. they do not include k other than in product with some large parameter. Then, using eq. (268) we can write the kinetic potential as the Taylor series expansion

$$V(y,B) = \sum_{n=1}^{\infty}\frac{1}{m!}\beta^{m-1}\sum_{\alpha_1\ldots\alpha_n}\Lambda^{\alpha_1\ldots\alpha_n}(B)y_{\alpha_1}\ldots y_{\alpha_n}, \quad \text{where} \quad \beta = \frac{1}{kT}. \tag{269}$$

Therefore, the Markov generating function and the kinetic potential are related by comparing eqs. $(258,269)$

$$\Phi(v,B) = \beta V(kTV,B), \tag{270}$$

so that the following master equation

$$\frac{\partial p(B)}{\partial \tau} = N_{\partial,B}\beta V\Big(-kT\frac{\partial}{\partial B},B\Big)p(B), \tag{271}$$

is valid.

Let us introduce the free energy function $F(B)$ by

$$p_{\text{eq}}(B) = \exp(-\beta F(B)), \tag{272}$$

i.e.

$$\psi^2(B) = \exp(-\beta F(B)). \tag{273}$$

We introduce the probability density

$$p_z(b) = Cexp(-\beta[F(B) - z.B]). \tag{274}$$

Here we have introduced hypothetical thermodynamic forces $z = (z^\alpha)$ which couple to the fluctuational processes B, through the internal product $z.B = z_\alpha B^\alpha$. Substituting (259) into eq. (269), we obtain

$$V(y,B) = \lim_{\tau\to 0}\sum_{m=1}^{\infty}\frac{\beta^{m-1}}{m!} < (y_\alpha \triangle B^\alpha)^m > \; = \frac{1}{\beta}\lim_{\tau\to 0} < \exp(\beta y_\alpha \triangle B^\alpha) - 1 >_B .$$

Let us compute the matrix of second derivatives of V:

$$V_{\alpha\beta} := \frac{\partial^2 V(y,B)}{\partial y^\alpha \partial y^\beta} = \beta\lim_{\tau\to 0}\frac{1}{\tau} < \triangle B^\alpha \triangle B^\beta \exp(\beta y \triangle B) >_B . \tag{275}$$

let us prove that this matrix is positive definite: $V_{\alpha\beta}a^\alpha a^\beta \geq 0$, for any vector $a = (a^\alpha)$. Indeed, by eq. (276)

$$V_{\alpha\beta}a_\alpha a_\beta = \beta\lim_{\tau\to 0}\frac{1}{\tau} < (\triangle B^\alpha)^2 \exp(\beta y \triangle B) >_B \geq 0. \tag{276}$$

Consequently, $V(y,B)$ is a convex function of y. From the convexity of V with respect to y, we conclude that the kinetic potential of the fluctuations is non-negative:

$$V_{\text{fluc}}(y,B) := V(y,B) - \Lambda^\alpha(B)y_\alpha = \sum_{m=2}^{\infty}\frac{\beta^{m-1}}{m!}\Lambda^{\alpha_1\ldots\alpha_n}y_{\alpha_1}\ldots y_{\alpha_n} \geq 0. \tag{277}$$

We shall now proceed to introduce the *image* of the kinetic potential, $R(y,z)$, which is defined by

$$R(y,z) := \int V(y,B)p_z(B). \tag{278}$$

Consider next the equilibrium density $\rho_{\text{equil}} = \psi^2$, i.e. the time-independent solution of the Fokker-Planck equation for a RCW geometry with trace-torsion given by $\frac{1}{2}d\ln\rho_{equil}$. We naturally enquire what happens when we rescaled ρ_{equil} to p_z. The obvious fact is that the operator of multiplication by $\exp(\beta x.B) = \exp(\beta x_\alpha B^\alpha)$ does not commute with $\frac{\partial}{\partial B}$. Instead we have

$$\exp(\beta z.B) f\Big(\frac{\partial}{\partial B}\Big)\exp(\beta z.B). \tag{279}$$

Then, we have

$$\exp(\beta z.B)N_{\partial B}V\Big(-kT\frac{\partial}{\partial B}\Big) = N_{\partial B}V\Big(-kT\frac{\partial}{\partial B} + z.B\Big)\exp(\beta z.B). \tag{280}$$

Inserting this in the Fokker-Planck equation multiplied by $\exp(\beta x.B)$ and further integrating with respect to B on all space, we find that all terms that have $\frac{\partial}{\partial B}$ in the argument of V vanish. Thus the integral that remains is

$$\int V(z,B)\exp(\beta z.B)\rho_{\text{equil}}(B)dB = 0, \tag{281}$$

i.e.

$$< V(x,B) >_z = 0, \quad \text{with} \quad < f >_z = \int f(B)\rho_{\text{equil}} dB, \tag{282}$$

or still,

$$R(z,z) = 0, \forall z. \tag{283}$$

Due to the smallness of the fluctuations of B, the probability density is concentrated near its maximum point $A(z)$. Therefore, from the definition of the image of the kinetic potential, we obtain the approximate expression

$$R(y,z) = V(y,A(z)) + O(k). \tag{284}$$

This will have profound implications in the formulation of the non-linear Boltzmann H-theorem. Substituting the eq. (259) in eq. (269) we get the expansion

$$R(y,z) = \sum_{m=1}^{\infty} \frac{\beta^{m-1}}{m!} \kappa^{\alpha_1 \ldots \alpha_m}(z) y_{\alpha_1} \ldots y_{\alpha m}, \tag{285}$$

where

$$\kappa^{\alpha_1 \ldots \alpha_m}(z) = \int \Lambda^{\alpha_1 \ldots \alpha_m}(B)\rho_z(B) dB. \tag{286}$$

We note that from eqs.$(269, 278)$ we obtain the expression

$$\kappa^{\alpha_1 \ldots \alpha_m}(z) = \lim_{\tau \to 0} \frac{1}{\tau} < \triangle B^{\alpha_1} \ldots \triangle B^{\alpha_m} >_z . \tag{287}$$

Substituting eq. (264) into eq. (287) we obtain the formula analogous to eq. (275):

$$R(y,z) = kT \lim_{\tau \to 0} < \exp(\beta y.B) - 1 >_z . \tag{288}$$

Analogously as the proof for the convexity of V, one can prove that $R(y,z)$ is convex as a function of the variable y, and $R_{\text{fluc}}(y,z)$, the fluctuational part of $R(y,z)$ defined by

$$R_{\text{fluc}}(y,z) = R(y,z) - \kappa^{\alpha} y_{\alpha}, \tag{289}$$

verifies

$$R_{\text{fluc}}(y,z) \geq 0. \tag{290}$$

It is immediate that the image of the kinetic potential admits the integral representation derived from the canonical representation, given by

$$R(y,z) = \kappa^{\alpha}(z) y_{\alpha} + \int_{-\infty}^{+\infty} f(s) s^{-2} G(s,z) ds, \tag{291}$$

where

$$G(s,z) = \int H(kTs,B)\rho_z(B) dB. \tag{292}$$

17. The Geometry of the Non-linear H-Theorem

In this following we shall assume a fluctuational homogeneous Markov process described by the stochastic Itô differential equation

$$dB_\tau = b(B_\tau)d\tau + \sigma(B_\tau)dW_\tau, \tag{293}$$

where the drift vector field $b = b^\alpha \frac{\partial}{\partial x^\alpha}$ has for components

$$b = \operatorname{grad}\ln\psi + \frac{1}{\psi^2} g^{\alpha\beta}(\delta\beta_2 + \omega_1)_\beta, \tag{294}$$

which is the g-conjugate (i.e. $b^\alpha = g^{\alpha\beta}Q_\beta$) to the differential one-form

$$Q = d\ln\psi + \frac{1}{\psi^2}(\delta\beta_2 + \omega_1), \tag{295}$$

and the noise tensor σ verifies that $\sigma\sigma^\dagger = g$, where g is a positive-definite metric or Minkowski. We recall that

$$\Lambda^\alpha = b^\alpha, \quad \Lambda^{\alpha\beta} = g^{\alpha\beta}, \quad \rho_{\text{equil}} = e^{-kF}, \tag{296}$$

where F is the free-energy function. We further have

$$\kappa^\alpha(z) = < b^\alpha >_z = \int b^\alpha(B)\psi^2(B)\exp(\beta z.B)dB. \tag{297}$$

Assume that the means

$$A^\alpha(\tau) = < B^\alpha_\tau >, \tag{298}$$

satisfy the non-linear fluctuation-dissipation equations

$$\frac{dA^\alpha}{d\tau} = < dB_\tau > = \phi^\alpha(A(\tau)). \tag{299}$$

Yet, since

$$\frac{d < B_\tau >}{d\tau} = < dB_\tau > = b(B_\tau), \tag{300}$$

where the first identity follows from the Itô equation and the second identity follows from eq. (293), the fluctuation-dissipation equations are

$$b(B_\tau) = \phi(A(\tau)), \quad \text{where } A(\tau) = < B_\tau > . \tag{301}$$

It is remarkable that in this last form, the fluctuation-dissipation equations have *no* relations with proper-time *evolution* of the fluctuation's means.

Now from eq. (287) for any $m \geq 1$ we get

$$< dB_\tau >_z = \kappa^\alpha(A). \tag{302}$$

Yet, since $p_z(B)$ has a maximum at $A(z)$, we have that

$$< dB^\alpha_\tau >_z = < \phi^\alpha(A) >_z = \phi^\alpha(A(z)), \tag{303}$$

and therefore, the fluctuation-dissipation equations assume the form

$$\frac{dA^\alpha}{d\tau} = \phi^\alpha(A(z)) = \kappa^\alpha(z(A)). \tag{304}$$

Finally, rewriting the non-evolutionary form eq. (302) of the fluctuation-dissipation equations we finally obtain

$$b^\alpha(B_\tau) = \kappa^\alpha(z). \tag{305}$$

The relation between z and A is that the variables z are assumed to be thermodynamic forces conjugate to A, i.e.

$$z = \nabla_A F = \frac{\partial F(A)}{\partial A}, \tag{306}$$

where since F is a scalar field, then the gradient ∇ defining the thermodynamic forces can be either the usual partial derivative or the covariant derivative either with respect to the metric (the Levi-Civita covariant derivative) or with respect to the full connection with torsion given by the g-conjugate of b, so in the definition of z we can neglect the self-interaction with the geometry of the Markov process. Therefore, from eq. (302) we conclude that

$$R(z,z) = z_\alpha \kappa^\alpha(z) + R_{\text{fluc}}(z,z) = 0, \tag{307}$$

then, from eq. (290) we conclude that

$$z_\alpha \kappa^\alpha(z) = -R_{\text{fluc}}(z,z) \leq -R(y,z) \leq 0, \tag{308}$$

so that

$$z_\alpha \kappa^\alpha(z) \leq 0. \tag{309}$$

From eqs. (299, 309) and the chain rule for derivative, follows that

$$\frac{dF(A)}{d\tau} = \frac{\partial F(A)}{\partial A}\frac{dA}{d\tau} = z_\alpha \frac{dA^\alpha}{d\tau} = z_\alpha \kappa^\alpha(z) \leq 0. \tag{310}$$

Therefore,

$$\frac{dF(A)}{d\tau} \leq 0, \tag{311}$$

which from eqs. (273, 298) can be further written in terms of the torsion-drift as

$$-\beta \frac{d \ln \psi^2(< B_\tau >)}{d\tau} \leq 0. \tag{312}$$

From eqs. (311, 287, 289) it follows that the derivative with respect to proper-time of the free energy

$$\frac{dF(A)}{d\tau} = -kT \lim_{\tau\to 0} \frac{1}{\tau} < f(\beta x.B) >_z, \tag{313}$$

with $f(h) = \exp(h) - h - 1 > 0$ for $h \neq 0$, is strictly negative:

$$\frac{dF(A)}{d\tau} < 0, \tag{314}$$

and, by the first identity in eq. (310) and further using eqs. $(299,300)$, it vanishes in the case that

$$\frac{dA}{d\tau} = < dB_\tau > = b(B_\tau) = 0, \tag{315}$$

that is, it vanishes whenever the torsion-drift vectorfield along the random trajectories vanishes, or more generally, whenever b or still, if the metric g has no zeros, when the torsion one-form Q does vanish, due to eq. (238).

Therefore, the production of free energy, as a function of the averages A, strictly decreases whenever b or still Q does not vanish. In the particular case of four-dimensional spacetime, the previous condition establishes the existance of the Pfaffian sequence given by the differential forms $\{Q, dQ, Q\wedge dQ, dQ\wedge dQ\}$ and topological torsion can be associated and the topological dimension of four-dimensional spacetime domains where entropy decreases can be introduced. Thus, we conclude that the increase of the entropy for an isothermal process, $\frac{-1}{T}F$, is produced by the non-vanishing torsion, and the stationary entropy state is linked with a systemic syntropic action introduced by geometrical structures and processes associated to this null torsion. Of special interest is the case of identification of the origin and the point at infinity in the complex plane (and thus introducing the one-point compactification of this plane), when considering quantum jumps [76] produced by the *zeros* of ψ which produces a trace-torsion one-form $d\ln\psi$, with ψ either a complex or quaternionic-valued function on a Lorentzian manifold (i.e. provided with a degenerate metric g, say Minkowski metric) satisfying a wave propagation equation with respect to the Laplace-Beltrami operator $\triangle_g$, i.e.

$$\triangle_g \psi = 0, \quad \text{where} \quad \triangle_g = \mathrm{div_g grad_g} \tag{316}$$

and the non-linear eikonal equation for light-rays

$$g(d\psi, d\psi) \equiv (\nabla\psi)^2 = 0. \tag{317}$$

Furthermore, we have seen that the fluctuation-dissipation equation, which we finally write in the form

$$b(B_\tau) = \kappa\Big(\frac{\partial F}{\partial A}\Big), \quad \text{with} \quad A = < B_\tau >, \tag{318}$$

relates the torsion-drift evaluated along the stochastic flow equals its average with respect to the weighted density p_z, with $z = \frac{\partial F}{\partial A}$, i.e., one has to rescale the equilibrium density by the thermodynamic force produced by variation of the free energy defined by the equilibrium measure along the averaged stochastic flow further coupled to the stochastic flow. Thus, the fluctuation-dissipation equations appear as non-evolutionary, the time-τ-derivative does not appear at all in them, and furthermore, the *future* equilibrium states determine them in the sense already seen. The final equilibrium state of the fluctuations defined by the torsion geometry, is necessary for their formulation, in addition of the *integral* of the stochastic flow. Thus the history of the system given by the integration of the fluctuational process (i.e. B_τ) and the *final* conditions are fundamental to the growth of the entropy. Therefore, the time arrow associated to this entropy growth is produced by the *final* -stationary- conditions and its associated *non-null torsion*. We have modified for this the equilibrium density, i.e. modified the *final* state of the system by a density which incorporates the coupling of the stochastic flow to the thermodynamic forces given by the variation of the free energy defined by the equilibrium state with respect to the average of this same flow. In the case of a linear reversible drift vector field (i.e. one in which the electromagnetic terms are completely null, and one is only left with the gradient of the equilibrium density), this rescaling is almost completely tautological. This will be the subject of the next section.

18. Detailed Balance and the Onsager Reciprocity Relations of Linear Non-equilibrium Statistical Mechanics

We have seen that by rescaling the torsion geometrical structure in the non-linear case by introducing thermodynamical forces leads to the increase of entropy as long as the torsion one-form (which may include electromagnetic like components) does *not* vanish.

In this section, we shall show that as a consequence of assuming that the torsion is linear (be that for a torsion geometry defined on spacetime or thermodyamical state-space), the rescaling can be obviated since the local thermodynamical forces are represented, in this linear case, by the linear torsion itself. In other words, if the torsion is linear, this already establishes a time arrow without rescaling by thermodynamic forces, which are now linear as well by the said identification with the torsion, in this linear case.

This is precisely the case of the Ornstein-Uhlenbeck process [21, 90], also called the *harmonic oscillator* process. This process has a an equilibrium measure $\rho = \psi^2$ with ψ given by a (zero-mean) Gaussian function, and a constant metric and diffusion tensor as well.

Assume the components $Q_\alpha(x)$ of the trace-torsion are linear functions of x, i.e.

$$Q_\alpha(x) = C_{\alpha\beta}x_\beta, \quad \text{with} \quad C_{\alpha\beta} = \text{constant}, \ \forall\alpha,\beta, \tag{319}$$

and metric

$$g_{\alpha\beta}(x) = B_{\alpha\beta} = \text{constant}, \quad \forall\alpha,\beta. \tag{320}$$

The Markov process is described then by the linear Itô stochastic differential equation

$$dB_\tau = CB_\tau + DdW_\tau, \tag{321}$$

where $C = (C_{\alpha\beta})$ and D is a matrix with constant entries such that

$$DD^\dagger = B. \tag{322}$$

The detailed balance equations are

$$(\varepsilon_\alpha\varepsilon_\beta C_{\alpha\beta} + C_{\alpha\beta})x_\beta = \frac{\partial \ln \rho_{\text{equil}}(x)}{\partial x^\alpha}, \tag{323}$$

and

$$\varepsilon_\alpha\varepsilon_\beta B_{\alpha\beta} = B_{\alpha\beta}. \tag{324}$$

The first condition implies that $\rho_{\text{equil}}(x)$ is a Gaussian function, since $d\ln\psi$ is assumed to be linear on x, and still, since eq. (323) does not contain an extra constant coefficient, then the Gaussian is of zero mean, i.e.

$$\rho_{\text{equil}} = \psi^2(x) = \exp\left(\frac{-1}{2}x^\dagger\tilde{\sigma}^{-1}x\right). \tag{325}$$

After rearranging and using the symmetry of $\tilde{\sigma}$, the stationary Fokker-Planck equation yields

$$-C_{\alpha\alpha} - \frac{1}{2}B_{\alpha\beta}\tilde{\sigma}^{-1}_{\alpha\beta} + \Big(\tilde{\sigma}^{-1}_{\kappa\alpha}C_{\alpha\beta} + \frac{1}{2}\tilde{\sigma}^{-1}_{\kappa\alpha}B_{\alpha\gamma}\tilde{\sigma}^{-1}_{\gamma\beta}\Big)x_\kappa x_\beta = 0. \tag{326}$$

Yet, the quadratic term given by the sum of the second and third terms in eq. (326) vanishes if the symmetric part of its coefficient is zero. We may write this condition in matrix form as

$$\tilde{\sigma}^{-1}C + C^\dagger\tilde{\sigma}^{-1} = -\tilde{\sigma}^{-1}B\tilde{\sigma}^{-1}, \tag{327}$$

or still,

$$C\tilde{\sigma} + \tilde{\sigma}C^\dagger = -B. \tag{328}$$

Thus, if this is the case, from eq. (326) we have that

$$C_{\alpha\alpha} = 0, \forall\alpha. \tag{329}$$

Let us introduce now the square matrix $\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots)$, with $\varepsilon_i = \pm 1, i = 1, \ldots, n$. Thus

$$\varepsilon^2 = I, \tag{330}$$

the identity matrix. Then, the detailed balance conditions can be rewritten as

$$\varepsilon C\varepsilon + C = -B\tilde{\sigma}^{-1}, \tag{331}$$

and

$$\varepsilon B\varepsilon = B. \tag{332}$$

Yet, detailed balance further requires that

$$\varepsilon\tilde{\sigma} = \tilde{\sigma}\varepsilon. \tag{333}$$

Multiplying eq. (328) by $\tilde{\sigma}$ takes the form

$$C\tilde{\sigma} + \tilde{\sigma}C = -B \tag{334}$$

and then

$$\varepsilon C\varepsilon\tilde{\sigma} = \tilde{\sigma}C^\dagger, \tag{335}$$

which with eq. (328) finally yields

$$\varepsilon(C\tilde{\sigma}) = (C\tilde{\sigma})^\dagger\varepsilon. \tag{336}$$

These are the reknown Onsager fluctuation-dissipation equations of linear non-equilibrium statistical mechanics [21,58,64]. Since C is the linear coefficient matrix of the trace-torsion, and $\tilde{\sigma}$ is the symmetric matrix defining the torsion-drift of the equilibrium density, we note that they deal with torsion. In fact, the interpretation with respect to a fluctuation-dissipation can be better understood by introducing a force given by minus the torsion-drift vectorfield corresponding to the Gaussian function $\rho_{\mathrm{equil}}(x) = \psi^2 = \exp(\frac{-1}{2}x^\dagger\tilde{\sigma}x)$, i.e., we take

$$F(x) = -\mathrm{grad}\ln\rho = \tilde{\sigma}^{-1}x. \tag{337}$$

Here, $F(x)$ is the free energy of the system.

The fluctuation-dissipation relations are the phenomenological equations

$$\frac{d<x>}{d\tau} = \phi(<x>). \tag{338}$$

Whenever ϕ is an arbitrary function, then we are in the case of a non-linear fluctuation-dissipation relation, while the particular case of a linear function, ϕ, yields the linear fluctuation-dissipation eq. (336).

In the case of the Ornstein-Uhlenbeck process,

$$\phi(<x>) \equiv b(<x>) = C<x>, \tag{339}$$

which is equal to

$$C\tilde{\sigma}F(<x>) = LF(<x>), \text{ where } L = C\tilde{\sigma}. \tag{340}$$

Note thus that this is defined entirely in terms of the coefficients entering the definition of the torsion. To resume, if the fluctuation-dissipation is such that the fluxes $\frac{d<x>}{d\tau}$ are related linearly to the torsion-forces $F(<x>) = \frac{-1}{2}\text{grad}\ln\rho_{\text{equil}}$ by a matrix L defined by

$$F = L\tilde{\sigma}, \tag{341}$$

i.e.

$$\frac{d<x>}{d\tau} = F(<x>) = L<x>, \tag{342}$$

then, the fluctuation-dissipation relation given by eq. (336) yields

$$\varepsilon L\varepsilon = L^{\dagger}, \tag{343}$$

or,

$$L_{\alpha\beta} = L_{\alpha\beta}, \tag{344}$$

in case that $\varepsilon_{\alpha}, \varepsilon_{\beta}$ have the same sign, and instead

$$L_{\alpha\beta} = -L_{\beta\alpha}, \tag{345}$$

in case that $\varepsilon_{\alpha}, \varepsilon_{\beta}$ are of the different sign. These are the Onsager reciprocity relations written in their usual form. Notice also that

$$\varepsilon B\varepsilon = B, \tag{346}$$

and

$$\varepsilon\tilde{\sigma}\varepsilon = \tilde{\sigma}, \tag{347}$$

which imply that $B_{\alpha\beta}$ and $\tilde{\sigma}_{\alpha\beta}$ vanish if ε_{α} and ε_{β} have opposite signs. Note that the former means the vanishing of the metric, while the latter the vanishing of the drift, and thus in the case that there is a time-reversal (the opposite signs of ε), the torsion does indeed vanish and thus the entropy does *not* grow. In the present case of the Uhlenbeck-Orstein process, the irreversible components of the trace torsion are by assumption inexistent, so that detailed balance is present. Notwithstanding this, the reversible components of the trace-torsion give the linear fluctuation-dissipation relations of linear non-equilibrium statistical mechanics, and still, when we have a change of sign due to detailed balance, the entropic reversibility is guaranteed at the origin $x = 0$ or in the zeros of $\tilde{\sigma}$, the phase factor of ψ^2.

Observations. The conclusions that we have reached above, namely that the growth of the entropy is related to non-zero drift (which we recall arises as the g-conjugate of the trace-torsion, $b^{\alpha} = g^{\alpha\beta}Q_{\beta}$), and that this is the origin of the time-arrow, were derived first in [66]. These conclusions stand in deep contrast with the tenants that stemmed from Prigogine's pioneering work in linear non-equilibrium thermodynamics, in which the the growth of the entropy is related to "internal" fluctuations of the system which are unrelated in that work to geometrical structures [64], and unconnected to scales in Nature. Indeed, what makes the fluctuations "internal" microscopic and the macroscopic character of the system are the precision of scales in which "microscopic" and "macroscopic" become defined. The non-linear theory due to Stratonovich stems from similar notions and arise from hamiltonians that are the natural structure of conservative systems [90], yet we have kept his constructions -yet not only circumscribed to state space but yet extended to spacetime itself. Further, our constructions do not start as in the usual approaches from hamiltonians but from geometrical constructs. Furthermore, it has been proved that the action of dissipative systems drive to non-equilibrium originally equilibrium systems representable by a hamiltonian in such a way that the Gibbs entropy diverges to minus infinity, rather than growing nor becoming stationary,the latter case we proved to be produced by the vanishing of the drift vector field b derived from the torsion Q acted by the metric g(which, we recall, represents the covariance of the fluctuations, since $b^{\alpha} = g^{\alpha\beta}Q_{\beta} = \Lambda^{\alpha\beta}Q_{\beta}$, from eqs. $(237,238)$). Thus we are lead to query how is it that in the present approach we can have an actual reduction of entropy. The answer to this was provided by Stratonovich, by noting that in the case in which there is no heat exchange between the system and its environment so that the entropy of the averages, $S(A(\tau))$ where $A(\tau) =< B(\tau >)$, must not decrease as we saw already, whilst the *random* entropy, $S(B_{\tau})$ can fluctuate and decrease by a quantity of the order k; see page 44 in Stratonovich [90]. In the case studied by Evans and Lamberti [16,93], the equilibrium system does dissipate heat through the boundaries by the action of forces that drives it to a non-equilibrium state, yet while the Gibbs entropy of the non-equilibrium system decreases to minus infinity, the Gibbs entropy of the whole system, the non-equilibrium driven system and the environment, remains constant. So in this case we see a similar situation than the one envisaged by Stratonovich: while the microscopic system reduces the entropy, the total entropy of the macroscopic system complies with the second law. This requires, at least, two scales, defining micro and macro. In our approach we have also turned to the averages $A(\tau) =< B(\tau) >$ and the fluctuation-dissipation relations are non-evolutionary and the second-law of thermodynamics applies to the averages while the entropy of the fluctuations can decrease initiating thus a syntropic process which already when the entropy of the averages becomes stationary and in the case of dimension equal to 4 we are lead to hamiltonian systems [33] whenever the drift b vanishes. Yet, the derivation of the non-linear H-theorem has relied in the fact that the evolution of the means, $\frac{d<B(\tau)>}{d\tau} \equiv \frac{dA(\tau)}{d\tau}$ is equal to $b(B(\tau))$ (where the first identity follows from the Ito formula [21,30,46] and the second identity follows from eq. (300)). To recapitulate, we have abandoned the continuous non-differentiable fluctuations to consider their averages to study their evolution and the non-decrease of the entropy, to find that there is no real time-evolution since the time-derivative of the averages of the fluctuations is coded into the drift along the fluctuations *yet without taking their averages* So there is no time-evolution at all with respect to the averages of the fluctuations, and no time-arrow associated to the growth of the entropy arises unless one abandons the fluctuations and goes to their averages, and yet the information that defines the time arrow is coded by the drift along the fluctuations, the term $b(B(\tau))$. In the other hand the random entropy $S(B(\tau))$ defined by the fluctuations (i.e. without considering their averages) can be negative and still the second law of thermodynamics is valid.

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INTERSECTING FAMILIES OF SETS AND PERMUTATIONS: A SURVEY

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Abstract

A family $\mathcal{A}$ of sets is said to be *t-intersecting* if any two sets in $\mathcal{A}$ have at least t common elements. A central problem in extremal set theory is to determine the size or structure of a largest t-intersecting sub-family of a given family $\mathcal{F}$. We give a survey of known results, conjectures and open problems for various important families $\mathcal{F}$, namely, power sets, levels of power sets, hereditary families, families of signed sets, families of labeled sets, and families of permutations. We also provide some extensions and consequences of known results.

1. Introduction

Unless otherwise stated, we shall use small letters such as x to denote elements of a set or non-negative integers or functions, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (i.e. sets whose elements are sets themselves). It is to be assumed that arbitrary sets and families are *finite*. We call a set A an *r-element set*, or simply an *r-set*, if its size $|A|$ is r (i.e. if it contains exactly r elements). A family is said to be *uniform* if all its sets are of the same size.

The set $\{1,2,\dots\}$ of positive integers is denoted by $\mathbb{N}$. For $m,n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ is denoted by $[m,n]$, and if $m = 1$ then we also write $[n]$. For a set X, the *power set* $\{A \colon A \subseteq X\}$ of X is denoted by 2^X, and the uniform sub-family $\{Y \subseteq X \colon |Y| = r\}$ of 2^X is denoted by $\binom{X}{r}$.

For a family $\mathcal{F}$ of sets, we denote the union of all sets in $\mathcal{F}$ by $U(\mathcal{F})$ and we denote the size of a largest set in $\mathcal{F}$ by $\alpha(\mathcal{F})$. For an integer $r \geq 0$, we denote the uniform sub-family $\{F \in \mathcal{F} \colon |F| = r\}$ of $\mathcal{F}$ by $\mathcal{F}^{(r)}$ (note that $\mathcal{F}^{(r)} = \binom{X}{r}$ if $\mathcal{F} = 2^X$), and we call $\mathcal{F}^{(r)}$ the *r'th level of* $\mathcal{F}$. For a set S, we denote $\{F \in \mathcal{F} \colon S \subseteq F\}$ by $\mathcal{F}(S)$. We may abbreviate $\mathcal{F}(\{x\})$ to $\mathcal{F}(x)$. If $x \in U(\mathcal{F})$ then we call $\mathcal{F}(x)$ a *star of* $\mathcal{F}$. More generally, if T is a t-element subset of a set in $\mathcal{F}$, then we call $\mathcal{F}(T)$ a *t-star of* $\mathcal{F}$.

A family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for any $A,B \in \mathcal{A}$. More generally, $\mathcal{A}$ is said to be *t-intersecting* if $|A \cap B| \geq t$ for any $A,B \in \mathcal{A}$. So an intersecting family is a 1-intersecting family. A t-intersecting family $\mathcal{A}$ is said to be *trivial* if $|\bigcap_{A \in \mathcal{A}} A| \geq t$ (i.e. there

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are at least t elements common to all the sets in $\mathcal{A}$); otherwise, $\mathcal{A}$ is said to be *non-trivial*. So a t-star of a family $\mathcal{F}$ is a trivial t-intersecting sub-family of $\mathcal{F}$ that is not contained in any other. If there exists a t-set T such that $\mathcal{F}(T)$ is a largest t-intersecting sub-family of $\mathcal{F}$ (i.e. no t-intersecting sub-family of $\mathcal{F}$ has more sets than $\mathcal{F}(T)$), then we say that $\mathcal{F}$ has the *t-star property at T*, or we simply say that $\mathcal{F}$ has the *t-star property*. If either $\mathcal{F}$ has no t-intersecting sub-families (which is the case if and only if $\alpha(\mathcal{F}) < t$) or all the largest t-intersecting sub-families of $\mathcal{F}$ are t-stars, then we say that $\mathcal{F}$ has the *strict* t-star property. We may abbreviate '1-star property' to 'star property'.

Extremal set theory is the study of how small or how large a system of sets can be under certain conditions. In this paper we are concerned with the following natural and central problem in this field.

Problem: *Given a family $\mathcal{F}$ and an integer $t \geq 1$, determine the size or structure of a largest t-intersecting sub-family of $\mathcal{F}$.*

We provide a survey of results that answer this question for families that are of particular importance, and we also point out open problems and conjectures. The survey papers [25] and [32] cover a few of the results we mention here and also go into many variations of the above problem; however, much progress has been made since their publication. Here we cover many of the important results that have been established to date, restricting ourselves to the problem stated above.

The most obvious families to consider are the power set $2^{[n]}$ and the uniform sub-family $\binom{[n]}{r}$, and in fact the problem for these families has been solved completely. However, there are other important families on which much progress has been made, and there are others that are still subject to much investigation. The families defined below are perhaps the ones that have received most attention and that we will be concerned with.

Hereditary families: A family $\mathcal{H}$ is said to be a *hereditary family* (also called an *ideal* or a *downset*) if all the subsets of any set in $\mathcal{H}$ are in $\mathcal{H}$. Clearly a family is hereditary if and only if it is a union of power sets. A *base* of $\mathcal{H}$ is a set in $\mathcal{H}$ that is not a subset of any other set in $\mathcal{H}$. So a hereditary family is the union of power sets of its bases. An example of a hereditary family is the family of independent sets of a *graph* or *matroid*.

Families of signed sets: Let X be an r-set $\{x_1,\dots,x_r\}$. Let $y_1,\dots,y_r \in \mathbb{N}$. We call the set $\{(x_1,y_1),\dots,(x_r,y_r)\}$ a *k-signed r-set* if $\max\{y_i : i \in [r]\} \leq k$. For an integer $k \geq 2$ we define $\mathcal{S}_{X,k}$ to be the family of k-signed r-sets given by

$$\mathcal{S}_{X,k} := \{\{(x_1,y_1),\dots,(x_r,y_r)\} : y_1,\dots,y_r \in [k]\}.$$

So a set A is a member of $\mathcal{S}_{X,k}$ if and only if it is a subset of the Cartesian product $X \times [k] := \{(x,y) : x \in X, y \in [k]\}$ satisfying $|A \cap (\{x\} \times [k])| = 1$ for all $x \in X$. We shall set $\mathcal{S}_{\emptyset,k} := \emptyset$. With a slight abuse of notation, for a family $\mathcal{F}$ we define

$$\mathcal{S}_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}.$$

Families of labeled sets: For $\mathbf{k} := (k_1,\dots,k_n)$ with $k_1,\dots,k_n \in \mathbb{N}$ and $k_1 \leq \dots \leq k_n$, we define the family $\mathcal{L}_{\mathbf{k}}$ of *labeled n-sets* by

$$\mathcal{L}_{\mathbf{k}} := \{\{(1,y_1),\dots,(n,y_n)\} : y_i \in [k_i] \text{ for each } i \in [n]\}.$$

Note that $\mathcal{S}_{[n],k} = \mathcal{L}_{(k_1,\dots,k_n)}$ with $k_1 = \cdots = k_n = k$.

An equivalent formulation for $\mathcal{L}_{\mathbf{k}}$ is the Cartesian product $[k_1] \times \cdots \times [k_n] := \{(y_1,\dots,y_n)\colon y_i \in [k_i] \text{ for each } i \in [n]\}$, but it is more convenient to work with n-sets than work with n-tuples (the alternative formulation demands that we change the setting of families of sets to one of sets of n-tuples).

For any $r \in [n]$, we define

$$\mathcal{L}_{\mathbf{k},r} := \left\{ \{(x_1, y_{x_1}),\dots,(x_r, y_{x_r})\}\colon \{x_1,\dots,x_r\} \in \binom{[n]}{r}, y_{x_i} \in [k_{x_i}] \text{ for each } i \in [r] \right\},$$

and we set $\mathcal{L}_{\mathbf{k},0} = \emptyset$. Thus, for any $0 \le r \le n$, $\mathcal{L}_{\mathbf{k},r}$ is the family of r-element subsets of the sets in $\mathcal{L}_{\mathbf{k}}$, and $\mathcal{L}_{\mathbf{k},n} = \mathcal{L}_{\mathbf{k}}$. We also define $\mathcal{L}_{\mathbf{k},\le r} := \bigcup_{i=0}^{r} \mathcal{L}_{\mathbf{k},i}$.

Families of permutations: For an r-set $X := \{x_1,\dots,x_r\}$, we define $\mathcal{S}^*_{X,k}$ to be the special sub-family of $\mathcal{S}_{X,k}$ given by

$$\mathcal{S}^*_{X,k} := \{\{(x_1,y_1),\dots,(x_r,y_r)\}\colon y_1,\dots,y_r \text{ are \textit{distinct} elements of } [k]\}.$$

Note that $\mathcal{S}^*_{X,k} \neq \emptyset$ if and only if $r \le k$. With a slight abuse of notation, for a family $\mathcal{F}$ we define $\mathcal{S}^*_{\mathcal{F},k}$ to be the special sub-family of $\mathcal{S}_{\mathcal{F},k}$ given by

$$\mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}^*_{F,k}.$$

An *r-partial permutation of a set N* is a pair (A, f) where $A \in \binom{N}{r}$ and $f\colon A \to N$ is an injection. An $|N|$-partial permutation of N is simply called a *permutation of N*. Clearly, the family of permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{[n],n}$, and the family of r-partial permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{\binom{[n]}{r},n}$.

Let X be as above. $\mathcal{S}^*_{X,k}$ can be interpreted as the family of permutations of sets in $\binom{[k]}{r}$: consider the bijection $\beta\colon \mathcal{S}^*_{X,k} \to \{(A,f)\colon A \in \binom{[k]}{r}, f\colon A \to A \text{ is a bijection}\}$ defined by $\beta(\{(x_1,a_1),\dots, (x_r,a_r)\}) := (\{a_1,\dots,a_r\}, f)$ where, for $b_1 < \cdots < b_r$ such that $\{b_1,\dots,b_r\} = \{a_1,\dots,a_r\}$, $f(b_i) := a_i$ for $i = 1,\dots,r$. $\mathcal{S}^*_{X,k}$ can also be interpreted as the sub-family $\mathcal{X} := \{(A,f)\colon A \in \binom{[k]}{r}, f\colon A \to [r] \text{ is a bijection}\}$ of the family of r-partial permutations of $[k]$: consider an obvious bijection from $\mathcal{S}^*_{X,k}$ to $\mathcal{S}^*_{\binom{[k]}{r},r}$ and another one from $\mathcal{S}^*_{\binom{[k]}{r},r}$ to $\mathcal{X}$.

2. Intersecting Sub-Families of $\binom{[n]}{r}$ and $2^{[n]}$

In this section we take t, r and n to be positive integers such that $t \le r \le n$.

The study of intersecting families took off with the publication of [28], which features the following classical result, known as the Erdős-Ko-Rado (EKR) Theorem.

Theorem 2.1 (EKR Theorem [28]). *If $r \le n/2$ and $\mathcal{A}$ is an intersecting sub-family of $\binom{[n]}{r}$, then $|\mathcal{A}| \le \binom{n-1}{r-1}$.*

This means that for $r \le n/2$, $\binom{[n]}{r}$ has the star property, because the bound $\binom{n-1}{r-1}$ is the size of any star of $\binom{[n]}{r}$. Note that if $r > n/2$, then any two r-element subsets of $[n]$ must intersect, and hence $\binom{[n]}{r}$ is an intersecting family (also note it is a non-trivial one, so $\binom{[n]}{r}$ does not have the star property in this case).

In order to prove Theorem 2.1, Erdős, Ko and Rado [28] introduced a method known as *compression* or *shifting*; see [32] for a survey on the uses of this powerful technique in extremal set theory. There are various proofs of Theorem 2.1, two of which are particularly short and beautiful: Katona's proof [40], which featured an elegant argument known as the *cycle method*, and Daykin's proof [22] using another fundamental result known as the Kruskal-Katona Theorem [41, 44]. Hilton and Milner [37] proved that for $r \le n/2$, the family $\mathcal{N}_{n,r} := \left\{A \in \binom{[n]}{r} \colon 1 \in A, A \cap [2, r+1] \neq \emptyset\right\} \cup \{[2, r+1]\}$ is a largest non-trivial intersecting sub-family of $\binom{[n]}{r}$, and since the size of $\mathcal{N}_{n,r}$ is $\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$, it follows that if $r < n/2$, then the stars of $\binom{[n]}{r}$ are the largest intersecting sub-families of $\binom{[n]}{r}$, i.e. $\binom{[n]}{r}$ has the strict star property. Note that if $r = n/2$, then any sub-family $\mathcal{A}$ of $\binom{[n]}{r}$ satisfying $|\mathcal{A} \cap \{A, [2r]\backslash A\}| = 1$ for all $A \in \binom{[n]}{r}$ is an intersecting sub-family of $\binom{[n]}{r}$ of size $\frac{1}{2}\binom{n}{r} = \frac{1}{2}\binom{2r}{r} = \binom{2r-1}{r-1}$, and hence one of maximum size (an example of such a family $\mathcal{A}$ is $\mathcal{N}_{2r,r}$, so $\binom{[n]}{r}$ does not have the strict star property if $r = n/2$).

Also in [28], Erdős, Ko and Rado initiated the study of t-intersecting families. They proved that for $t < r$, there exists an integer $n_0(r,t)$ such that for all $n \ge n_0(r,t)$, the largest t-intersecting sub-families of $\binom{[n]}{r}$ are the t-stars (which are of size $\binom{n-t}{r-t}$). For $t \ge 15$, Frankl [31] showed that the smallest such $n_0(r,t)$ is $(r-t+1)(t+1)+1$ and that if $n = (r-t+1)(t+1)$, then $\binom{[n]}{r}$ still has the t-star property but not the strict t-star property. Subsequently, using algebraic means, Wilson [58] proved that $\binom{[n]}{r}$ has the t-star property for any t and $n \ge (r-t+1)(t+1)$. Frankl [31] conjectured that among the largest t-intersecting sub-families of $\binom{[n]}{r}$ there is always at least one of the families $\left\{A \in \binom{[n]}{r} \colon |A \cap [t+2i]| \ge t+i\right\}$, $i = 0, 1, \dots, r-t$. A remarkable proof of this long-standing conjecture together with a complete characterisation of the extremal structures was finally obtained by Ahlswede and Khachatrian [1] by means of the *compression* technique introduced in [28].

Theorem 2.2 ([1]). *Let $\mathcal{A}$ be a largest t-intersecting sub-family of $\binom{[n]}{r}$.*
(i) *If $(r-t+1)(2+\frac{t-1}{i+1}) < n < (r-t+1)(2+\frac{t-1}{i})$ for some $i \in \{0\} \cup \mathbb{N}$ - where, by convention, $(t-1)/i = \infty$ if $i = 0$ - then $\mathcal{A} = \{A \in \binom{[n]}{r} \colon |A \cap X| \ge t+i\}$ for some $X \in \binom{[n]}{t+2i}$.*
(ii) *If $t \ge 2$ and $(r-t+1)(2+\frac{t-1}{i+1}) = n$ for some $i \in \{0\} \cup \mathbb{N}$, then $\mathcal{A} = \{A \in \binom{[n]}{r} \colon |A \cap X| \ge t+j\}$ for some $j \in \{i, i+1\}$ and $X \in \binom{[n]}{t+2j}$.*

It is worth mentioning that in [2] Ahlswede and Khachatrian went on to determine the largest non-trivial t-intersecting sub-families of $\binom{[n]}{r}$.

Erdős, Ko and Rado [28] pointed out the simple fact that $2^{[n]}$ has the star property (indeed, for any set A in an intersecting sub-family $\mathcal{A}$ of $2^{[n]}$, the complement $[n]\backslash A$ cannot be in $\mathcal{A}$, and hence the size of $\mathcal{A}$ is at most $\frac{1}{2}|2^{[n]}| = 2^{n-1}$, i.e. the size of a star of $2^{[n]}$); note that there are many non-trivial intersecting sub-families of $2^{[n]}$ of maximum size 2^{n-1} (such as $\{A \subseteq [n] \colon |A \cap [3]| \ge 2\}$), so $2^{[n]}$ does not have the strict star property. They also asked what the size of a largest t-intersecting sub-family of $2^{[n]}$ is for $t \geq 2$. The answer in a complete form was given by Katona [42].

Theorem 2.3 ([42]). *Let $t \geq 2$, and let $\mathcal{A}$ be a largest t-intersecting sub-family of* $2^{[n]}$.
(i) *If $n+t = 2l$ then $\mathcal{A} = \{A \subseteq [n] \colon |A| \geq l\}$.*
(ii) *If $n+t = 2l+1$ then $\mathcal{A} = \{A \subseteq [n] \colon |A \cap X| \geq l\}$ for some $X \in \binom{[n]}{n-1}$.*

It is interesting that for $n > t \geq 2$, $2^{[n]}$ does not have the t-star property.

Many other beautiful results were inspired by the seminal paper [28], as are the results we present in the subsequent sections.

3. Intersecting Sub-Families of Hereditary Families

Recall that $2^{[n]}$ has the star property. Also recall that the power set of a set X is the simplest example of a hereditary family since 2^X is a hereditary family with only one base (X). An outstanding open problem in extremal set theory is the following conjecture (see [14] for a more general conjecture).

Conjecture 3.1 ([19]). *If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ has the star property.*

Chvátal [20] verified this conjecture for the case when $\mathcal{H}$ is *left-compressed* (i.e. $\mathcal{H} \subseteq 2^{[n]}$ and $(H \backslash \{j\}) \cup \{i\} \in \mathcal{H}$ whenever $1 \leq i < j \in H \in \mathcal{H}$ and $i \notin H$). Snevily [54] took this result (together with results in [53, 55]) a significant step forward by verifying Conjecture 3.1 for the case when $\mathcal{H}$ is *compressed with respect to an element x of $U(\mathcal{H})$* (i.e. $(H \backslash \{h\}) \cup \{x\} \in \mathcal{H}$ whenever $h \in H \in \mathcal{H}$ and $x \notin H$).

Theorem 3.2 ([54]). *If a hereditary family $\mathcal{H}$ is compressed with respect to an element x of $U(\mathcal{H})$, then $\mathcal{H}$ has the star property at $\{x\}$.*

A generalisation is proved in [14] by means of an alternative self-contained argument. Snevily's proof of Theorem 3.2 makes use of the following interesting result of Berge [5] (a proof of which is also provided in [4, Chapter 6]).

Theorem 3.3 ([5]). *If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ is a disjoint union of pairs of disjoint sets, together with $\emptyset$ if $|\mathcal{H}|$ is odd.*

This result was also motivated by Conjecture 3.1 as it has the following immediate consequence.

Corollary 3.4. *If $\mathcal{A}$ is an intersecting sub-family of a hereditary family $\mathcal{H}$, then*

$$|\mathcal{A}| \leq \frac{1}{2}|\mathcal{H}|.$$

Proof. For any pair of disjoint sets, at most only one set can be in an intersecting family $\mathcal{A}$. By Theorem 3.3, the result follows. $\square$

A special case of Theorem 3.2 is a result of Schönheim [53] which says that Conjecture 3.1 is true if the bases of $\mathcal{H}$ have a common element, and this follows immediately from Corollary 3.4 and the following fact.

Proposition 3.5 ([53])**.** *If the bases of a hereditary family $\mathcal{H}$ have a common element x, then*
$$|\mathcal{H}(x)| = \frac{1}{2}|\mathcal{H}|.$$

Proof. Partition $\mathcal{H}$ into $\mathcal{A} := \mathcal{H}(x)$ and $\mathcal{B} := \{B \in \mathcal{H} \colon x \notin B\}$. If $A \in \mathcal{A}$ then $A \backslash \{x\} \in \mathcal{B}$; so $|\mathcal{A}| \leq |\mathcal{B}|$. If $B \in \mathcal{B}$ then $B \subseteq C$ for some base C of $\mathcal{H}$, and hence $B \cup \{x\} \in \mathcal{A}$ since $x \in C$; so $|\mathcal{B}| \leq |\mathcal{A}|$. Thus $|\mathcal{A}| = |\mathcal{B}| = \frac{1}{2}|\mathcal{H}|$. □

Many other results and problems have been inspired by Conjecture 3.1 or are related to it; see [21,51,57].

Conjecture 3.1 cannot be generalised to the t-intersection case. Indeed, if $n > t \geq 2$ and $\mathcal{H} = 2^{[n]}$, then by Theorem 2.3, $\mathcal{H}$ does not have the t-star property.

We now turn our attention to uniform intersecting sub-families of hereditary families, or rather intersecting sub-families of levels of hereditary families. For any hereditary family $\mathcal{H}$, let $\mu(\mathcal{H})$ denote the size of a smallest base of $\mathcal{H}$.

A *graph* G is a pair (V,E) with $E \subseteq \binom{V}{2}$, and a set $I \subseteq V$ is said to be an *independent set of* G if $\{i,j\} \notin E$ for any $i,j \in I$. Let $\mathcal{J}_G$ denote the family of all independent sets of a graph G. Clearly $\mathcal{J}_G$ is a hereditary family. Holroyd and Talbot [39] made a nice conjecture which claims that if G is a graph and $\mu(\mathcal{J}_G) \geq 2r$, then ${\mathcal{J}_G}^{(r)}$ has the star property, and ${\mathcal{J}_G}^{(r)}$ has the strict star property if $\mu(\mathcal{J}_G) > 2r$. In [11] the author conjectured that this is true for any hereditary family and that in general the following holds.

Conjecture 3.6 ([11])**.** *If $1 \leq t \leq r$, $\emptyset \neq S \subseteq [t,r]$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq (t+1)(r-t+1)$, then:*
(i) $\bigcup_{s \in S} \mathcal{H}^{(s)}$ *has the t-star property;*
(ii) $\bigcup_{s \in S} \mathcal{H}^{(s)}$ *has the strict t-star property if either $\mu(\mathcal{H}) > (t+1)(r-t+1)$ or $S \neq \{r\}$.*

Note that Theorem 2.2 solves the special case when $\mathcal{H} = 2^{[n]}$ and tells us that we cannot improve the condition $\mu(\mathcal{H}) \geq (t+1)(r-t+1)$. The author [11] proved that this conjecture is true if $\mu(\mathcal{H})$ is sufficiently large.

Theorem 3.7 ([11])**.** *Conjecture* 3.6 *is true if $\mu(\mathcal{H}) \geq (r-t)\binom{3r-2t-1}{t+1} + r$.*

The motivation behind establishing this result for any union of levels of a hereditary family $\mathcal{H}$ within a certain range is that this general form cannot be immediately deduced from the result for just one level of $\mathcal{H}$ (i.e. the case $S = \{r\}$ in Conjecture 3.6). As demonstrated in Example 1 in [11], the reason is simply that if T is a t-set such that $\mathcal{H}^{(s)}(T)$ ($s \in [t,r]$) is a largest t-star of the level $\mathcal{H}^{(s)}$, then for $p \neq s$ ($p \in [t,r]$), $\mathcal{H}^{(p)}(T)$ not only may not be a largest t-star of the level $\mathcal{H}^{(p)}$ but may be smaller than some non-trivial t-intersecting sub-family of $\mathcal{H}^{(p)}$. This is in fact one of the central difficulties arising from any EKR-type problem for hereditary families. In the proof of Theorem 3.7, this obstacle was overcome by showing that for any non-trivial t-intersecting sub-family $\mathcal{A}$ of the union, we can construct a t-star that is larger than $\mathcal{A}$ (and that is not necessarily a largest t-star). Many other proofs of EKR-type results are based on determining at least one largest t-star; as in the case of each theorem mentioned in Section 2., the setting is often symmetrical to the extent that all t-stars are of the same size and of a known size.

An interesting immediate consequence of Theorem 3.7 is that the union of the first $r \geq t$ levels of a hereditary family $\mathcal{H}$ has the strict t-star property if $\mu(\mathcal{H})$ is sufficiently larger than r.

Corollary 3.8 ([11]). *If $1 \leq t \leq r$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq (r-t)\binom{3r-2t-1}{t+1} + r$, then $\bigcup_{s=0}^{r} \mathcal{H}^{(s)}$ has the strict t-star property.*

Proof. Let $\mathcal{A}$ be a t-intersecting sub-family of $\bigcup_{s=0}^{r} \mathcal{H}^{(s)}$. Then no set in $\mathcal{A}$ is of size less than t, so $\mathcal{A} \subseteq \bigcup_{s \in S} \mathcal{H}^{(s)}$ with $S = [t,r]$. The result follows by Theorem 3.7. $\square$

This means that for the special case $t = 1$, we have the following.

Corollary 3.9 ([11]). *Conjecture 3.1 is true if $\mathcal{H} = \bigcup_{s=0}^{r} \mathcal{J}^{(s)}$ for some $r \in \mathbb{N}$ and some hereditary family $\mathcal{J}$ with $\mu(\mathcal{J}) \geq \frac{3}{2}(r-1)^2(3r-4) + r$.*

The following extension of Theorem 2.2 for $n \geq (t+1)(r-t+1)$ was also proved in [11].

Theorem 3.10 ([11]). *Conjecture 3.6 is true if $\mathcal{H}$ is left-compressed.*

4. Intersecting Families of Signed Sets

The ‘signed sets’ terminology was introduced in [10] for a setting that can be re-formulated as $\mathcal{S}_{\binom{[n]}{r},k}$, and the general formulation $\mathcal{S}_{\mathcal{F},k}$ was introduced in [13], the theme of which is the following conjecture.

Conjecture 4.1 ([13]). *For any family $\mathcal{F}$ and any $k \geq 2$,*
(i) *$\mathcal{S}_{\mathcal{F},k}$ has the star property;*
(ii) *$\mathcal{S}_{\mathcal{F},k}$ does not have the strict star property only if $k = 2$ and there exist at least three elements u_1, u_2, u_3 of $U(\mathcal{F})$ such that $\mathcal{F}(u_1) = \mathcal{F}(u_2) = \mathcal{F}(u_3)$ and $\mathcal{S}_{\mathcal{F},2}((u_1,1))$ is a largest star of $\mathcal{S}_{\mathcal{F},2}$.*

The converse of (ii) is true, and the proof is simply that $\{A \in \mathcal{S}_{\mathcal{F},2} \colon |A \cap \{(u_1,1),(u_2,1),(u_3,1)\}| \geq 2\}$ is a non-trivial intersecting sub-family of $\mathcal{S}_{\mathcal{F},2}$ that is as large as $\mathcal{S}_{\mathcal{F},2}((u_1,1))$.

In [14] a similarity between the intersection problem for hereditary families and the one presented above is demonstrated, and in fact a conjecture generalising both Conjecture 3.1 and the above conjecture is suggested.

Recall that a family $\mathcal{F}$ is said to be compressed with respect to an element x of $U(\mathcal{F})$ if $(F \backslash \{u\}) \cup \{x\} \in \mathcal{F}$ whenever $u \in F \in \mathcal{F}$ and $x \notin F$. The following is the main result in the paper featuring the above conjecture.

Theorem 4.2 ([13]). *Conjecture* 4.1 *is true if $\mathcal{F}$ is compressed with respect to an element x of $U(\mathcal{F})$, and $\mathcal{S}_{\mathcal{F},k}$ has the star property at $\{(x,1)\}$.*

Since $\binom{[n]}{r}$ is compressed with respect to any element of $[n]$, the above result has the following immediate consequence, which is a well-known result that was first stated by Meyer [50] and proved in different ways by Deza and Frankl [25], Bollobás and Leader [10], Engel [27] and Erdős et al. [29].

Theorem 4.3 ([10,25,27,29]). *Let $r \in [n]$ and let $k \geq 2$. Then:*
(i) $\mathcal{S}_{\binom{[n]}{r},k}$ *has the star property;*
(ii) *if $(r,k) \neq (n,2)$ then $\mathcal{S}_{\binom{[n]}{r},k}$ has the strict star property.*

Thus the size of an intersecting sub-family of $\mathcal{S}_{\binom{[n]}{r},k}$ is at most $\binom{n-1}{r-1}k^{r-1}$, i.e. the size of any star of $\mathcal{S}_{\binom{[n]}{r},k}$. Berge [6] and Livingston [49] had proved (i) and (ii), respectively, for the special case $\mathcal{F} = \{[n]\}$ (other proofs are found in [36,52]).

In [13] Conjecture 4.1 is also verified for the case when $\mathcal{F}$ is uniform and has the star property; Holroyd and Talbot [39] had essentially proved part (i) of the conjecture for such a family $\mathcal{F}$ in a graph-theoretical context.

The t-intersection problem for sub-families of $\mathcal{S}_{[n],k}$ has also been solved. Frankl and Füredi were the first to investigate it. In [33] they conjectured that among the largest t-intersecting sub-families of $\mathcal{S}_{[n],k}$ there is always one of the families $\mathcal{A}_i := \{A \in \mathcal{S}_{[n],k} \colon |A \cap ([t+2i] \times [1])| \geq t+i\}$, $i = 0,1,2,\ldots$, and they proved that if $k \geq t+1 \geq 16$, then $\mathcal{A}_0$ is extremal and hence $\mathcal{S}_{[n],k}$ has the star property. The conjecture was proved independently by Ahlswede and Khachatrian [3] and Frankl and Tokushige [34] (Kleitman [43] had long established this result for $k=2$). As in Theorem 2.2, Ahlswede and Khachatrian [3] also determined the extremal structures.

Theorem 4.4 ([3]). *Let $1 \leq t \leq n$ and $k \geq 2$. Let m be the largest integer such that $t+2m < \min\{n+1, t+2\frac{t-1}{k-2}\}$ (by convention, $\frac{t-1}{k-2} = \infty$ if $k=2$).*
(i) *If $(k,t) \neq (2,1)$ and $\frac{t-1}{k-2}$ is not integral, then $\mathcal{A}$ is a largest t-intersecting sub-family of $\mathcal{S}_{[n],k}$ if and only if*

$$\mathcal{A} = \{A \in \mathcal{S}_{[n],k} \colon |A \cap X| \geq t+m\}$$

for some $X \in \mathcal{S}_{Y,k}$ with $Y \in \binom{[n]}{t+2m}$.
(ii) *If $(k,t) \neq (2,1)$ and $\frac{t-1}{k-2}$ is integral, then $\mathcal{A}$ is a largest t-intersecting sub-family of $\mathcal{S}_{[n],k}$ if and only if*

$$\mathcal{A} = \{A \in \mathcal{S}_{[n],k} \colon |A \cap X| \geq t+j\}$$

for some $j \in \{m, m+1\}$ and some $X \in \mathcal{S}_{Y,k}$ with $Y \in \binom{[n]}{t+2j}$.
(iii) *If $(k,t) = (2,1)$, then $\mathcal{A}$ is a largest t-intersecting sub-family of $\mathcal{S}_{[n],k}$ if and only if for any $y_1,\ldots,y_n \in [2]$, exactly one of $\{(1,y_1),\ldots,(n,y_n)\}$ and $\{(1,3-y_1),\ldots,(n,3-y_n)\}$ is in $\mathcal{A}$.*

Note that (iii) follows trivially from the fact that for any set $A := \{(1,y_1),\ldots,(n,y_n)\}$ in $\mathcal{S}_{[n],2}$, $\{(1,3-y_1),\ldots,(n,3-y_n)\}$ is the only set in $\mathcal{S}_{[n],2}$ that does not intersect A. The rest of the theorem is highly non-trivial!

What led to Theorem 4.4 was the accomplishment of Theorem 2.2. The following is an immediate consequence of Theorem 4.4.

Corollary 4.5. *Let $1 \leq t \leq n$ and $k \geq 2$. Then:*
(i) *$\mathcal{S}_{[n],k}$ has the t-star property if and only if $k \geq t+1$;*
(ii) *$\mathcal{S}_{[n],k}$ has the strict t-star property if and only if $k \geq t+2$.*

We point out that Bey and Engel [9] extended Theorem 4.4 by determining the size of a largest non-trivial t-intersecting sub-family of $\mathcal{S}_{[n],k}$ (see Examples 10, 11 and Lemma 18 in [9]).

Note that $\mathcal{S}_{[n],k} = \mathcal{S}_{\binom{[n]}{r},k}$ with $r = n$. For the case $t \leq r < n$, Bey [8] proved the following.

Theorem 4.6 ([8]). *Let $1 \leq t \leq r < n$. $\mathcal{S}_{\binom{[n]}{r},k}$ has the t-star property if and only if $n \geq \frac{(r-t+k)(t+1)}{k}$.*

Thus, if $t \leq r < n$ and $n \geq \frac{(r-t+k)(t+1)}{k}$, then the size of a t-intersecting sub-family of $\mathcal{S}_{\binom{[n]}{r},k}$ is at most $\binom{n-t}{r-t}k^{r-t}$, i.e. the size of any t-star of $\mathcal{S}_{\binom{[n]}{r},k}$. From Corollary 4.5 and Theorem 4.6 we immediately obtain the following.

Corollary 4.7. *For any $1 \leq t \leq r \leq n$ and $k \geq t+1$, $\mathcal{S}_{\binom{[n]}{r},k}$ has the t-star property.*

To the best of the author's knowledge, no complete t-intersection theorem for $\mathcal{S}_{\binom{[n]}{r},k}$ has been obtained.

For the case when $\mathcal{F}$ is any family, the author [15] suggested the following general conjecture.

Conjecture 4.8 ([15]). *For any integer $t \geq 1$, there exists an integer $k_0(t)$ such that for any $k \geq k_0(t)$ and any family $\mathcal{F}$, $\mathcal{S}_{\mathcal{F},k}$ has the t-star property.*

In view of Corollary 4.7, we conjecture that the smallest $k_0(t)$ is $t+1$. In [15] it is actually conjectured that for some integer $k_0'(t)$, $\mathcal{S}_{\mathcal{F},k}$ has the strict t-star property for any $\mathcal{F}$, and hence, in view of Corollary 4.5(ii), we conjecture that the smallest $k_0'(t)$ is $t+2$. Note that Conjecture 4.1 claims that the smallest values of $k_0(1)$ and $k_0'(1)$ are 2 and 3, respectively. The author [15] proved the following relaxation of the statement of Conjecture 4.8.

Theorem 4.9 ([15]). *For any integers r and t with $1 \leq t < r$, let $k_0(r,t) := \binom{r}{t}\binom{r}{t+1}$. For any $k \geq k_0(r,t)$ and any family $\mathcal{F}$ with $\alpha(\mathcal{F}) \leq r$, $\mathcal{S}_{\mathcal{F},k}$ has the strict t-star property.*

The general idea behind the proof of this result is similar to that behind the proof of Theorem 3.7, described in Section 3..

Corollary 4.10. *Conjecture 4.1 is true if $k \geq \alpha(\mathcal{F})\binom{\alpha(\mathcal{F})}{2}$.*

5. Intersecting Families of Labeled Sets

Consider the family $\mathcal{L}_{\mathbf{k}}$, $\mathbf{k} = (k_1, \ldots, k_n)$, of *labeled n-sets*. If $k_1 = 1$ then all the sets in $\mathcal{L}_{\mathbf{k}}$ contain the point $(1,1)$ and hence $\mathcal{L}_{\mathbf{k}}$ has the strict star property. Berge [6] proved that for any $\mathbf{k}$, $\mathcal{L}_{\mathbf{k}}$ has the star property, and hence the size of an intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$ is at most the size $\frac{1}{k_1}|\mathcal{L}_{\mathbf{k}}| = k_2k_3\ldots k_n$ of the star $\mathcal{L}_{\mathbf{k}}((1,1))$, as this is clearly a largest star (since $k_1 \leq \cdots \leq k_n$). We shall reproduce the remarkably short proof of this result.

Let mod* be the usual modulo operation with the exception that for any integer a, a mod* a is a instead of 0. For any integer q, let $\theta_{\mathbf{k}}^q : \mathcal{L}_{\mathbf{k}} \to \mathcal{L}_{\mathbf{k}}$ be the *translation operation* defined by

$$\theta_{\mathbf{k}}^q(A) := \{(a, (b+q) \bmod^* k_a) \colon (a,b) \in A\},$$

and define $\Theta_{\mathbf{k}}^q : 2^{\mathcal{L}_{\mathbf{k}}} \to 2^{\mathcal{L}_{\mathbf{k}}}$ by

$$\Theta_{\mathbf{k}}^q(\mathcal{F}) := \{\theta_{\mathbf{k}}^q(A) \colon A \in \mathcal{F}\}.$$

Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$. For any $A \in \mathcal{A}$ and $q \in [k_1 - 1]$, we have $\theta_{\mathbf{k}}^q(A) \cap A = \emptyset$ and hence $\theta_{\mathbf{k}}^q(A) \notin \mathcal{A}$. Therefore $\mathcal{A}, \Theta_{\mathbf{k}}^1(\mathcal{A}), \dots, \Theta_{\mathbf{k}}^{k_1-1}(\mathcal{A})$ are k_1 disjoint sub-families of $\mathcal{L}_{\mathbf{k}}$. So $k_1|\mathcal{A}| \leq |\mathcal{L}_{\mathbf{k}}|$ and hence $|\mathcal{A}| \leq \frac{1}{k_1}|\mathcal{L}_{\mathbf{k}}|$.

Livingston [49] proved that for $3 \leq k_1 = \cdots = k_n$, $\mathcal{L}_{\mathbf{k}}$ has the strict star property. Using the shifting technique (see [32]) in an inductive argument, the author [12] extended Livingston's result for the case when $3 \leq k_1 \leq \cdots \leq k_n$. The above results sum up as follows.

Theorem 5.1 ([6,12,49])**.** *Let $1 \leq k_1 \leq \cdots \leq k_n$ and let $\mathbf{k} := (k_1, \dots, k_n)$. Then:*
(i) *$\mathcal{L}_{\mathbf{k}}$ has the star property at $\{(1,1)\}$;*
(ii) *if $k_1 \neq 2$ then $\mathcal{L}_{\mathbf{k}}$ has the strict star property.*

If $k_1 = 2$ then $\mathcal{L}_{\mathbf{k}}$ may not have the strict star property; indeed, if $k_1 = k_2 = k_3$ then $\{A \in \mathcal{L}_{\mathbf{k}} \colon |A \cap \{(1,1),(2,1),(3,1)| \geq 2\}$ is a non-trivial intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$ whose size is $\frac{1}{k_1}|\mathcal{L}_{\mathbf{k}}|$ (i.e. the maximum).

Recall that $\mathcal{S}_{[n],k} = \mathcal{L}_{(k_1,\dots,k_n)}$ with $k_1 = \cdots = k_n = k$. The same argument used in [12] to extend Livingston's result [49] gives the following extension of part (the sufficiency conditions) of Corollary 4.5 and generalisation of Theorem 5.1 with $k_1 \geq 2$.

Theorem 5.2. *Let $2 \leq t+1 \leq k_1 \leq \cdots \leq k_n$ and let $\mathbf{k} := (k_1, \dots, k_n)$. Then:*
(i) *$\mathcal{L}_{\mathbf{k}}$ has the t-star property at $\{(1,1), \dots, (t,1)\}$;*
(ii) *if $k_1 \geq t+2$ then $\mathcal{L}_{\mathbf{k}}$ has the strict t-star property.*

As we can see from Theorem 4.4 and Corollary 4.5, $\mathcal{L}_{\mathbf{k}}$ may not have the t-star property when $2 \leq k_1 \leq t$. Recall that for the case $k_1 = \cdots = k_n$, the extremal structures are given in Theorem 4.4, and they are all non-trivial when $2 \leq k_1 \leq t$.

The intersection problem for the families $\mathcal{L}_{\mathbf{k},r}$, $r = 1, \dots, n$, has also been treated to a significant extent. Note that $\mathcal{S}_{\binom{[n]}{r},k} = \mathcal{L}_{(k_1,\dots,k_n),r}$ with $k_1 = \cdots = k_n = k$. Using the shifting technique (see [32]) in an inductive argument, Holroyd, Spencer and Talbot [38] extended Theorem 4.3(i) as follows.

Theorem 5.3 ([38])**.** *Let $2 \leq k_1 \leq \cdots \leq k_n$ and let $\mathbf{k} := (k_1, \dots, k_n)$. Then for any $r \in [n]$, $\mathcal{L}_{\mathbf{k},r}$ has the star property at $\{(1,1)\}$.*

The proof of their result can be easily extended to obtain that $\mathcal{L}_{\mathbf{k},r}$ has the strict star property if $(r, k_1) \neq (n, 2)$ (see, for example, the proof of [12, Theorem 1.4]). The case $k_1 = 1$ proved to be harder, and Bey [7] solved it by applying the idea of *generating sets* introduced in [1].

Theorem 5.4 ([7])**.** *Let $1 = k_1 = \cdots = k_m < k_{m+1} \leq \cdots \leq k_n$ and let $\mathbf{k} := (k_1, \dots, k_n)$. Let $p := \lfloor (m+1)/2 \rfloor$, and for each $i \in [p]$, let $\mathcal{A}_i := \{A \in \mathcal{L}_{\mathbf{k},r} \colon (1,1) \in A, i \leq |A \cap \{(1,1), \dots, (m,1)\}| \leq m-i\} \cup \{A \in \mathcal{L}_{\mathbf{k},r} \colon |A \cap \{(1,1), \dots, (m,1)\}| \geq m-i+1\}$. Then one of the families $\mathcal{A}_1, \dots, \mathcal{A}_p$ is a largest intersecting sub-family of $\mathcal{L}_{\mathbf{k},r}$.*

Bey [7] also showed that when $r \leq n/2$ in the above theorem, $\mathcal{L}_{\mathbf{k},r}$ has the star property at $(1,1)$ (this is also proved in [38], and in [16] it is shown that $\mathcal{L}_{\mathbf{k},r}$ has the strict star property if $r < n/2$).

For the case when k_1 can be any positive integer but n is sufficiently large, Theorem 3.7 gives us the following t-intersection result.

Theorem 5.5. *Let $1 \leq t \leq r$ and let $n \geq (r-t)\binom{3r-2t-1}{t+1} + r$. Let $1 \leq k_1 \leq \cdots \leq k_n$ and let $\mathbf{k} := (k_1, \ldots, k_n)$. Then:*
(i) *$\mathcal{L}_{\mathbf{k},r}$ has the t-star property at $\{(1,1), \ldots, (t,1)\}$.*
(ii) *$\mathcal{L}_{\mathbf{k},r}$ has the strict t-star property.*

Proof. Let $\mathcal{H} := \mathcal{L}_{\mathbf{k},\leq n}$. Then clearly $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) = n$. Thus, by Theorem 3.7 (with $S = \{r\}$), $\mathcal{H}^{(r)}$ has the strict t-star property. Part (ii) follows since $\mathcal{H}^{(r)} = \mathcal{L}_{\mathbf{k},r}$. This in turn proves (i) since the family $\mathcal{L}_{\mathbf{k},r}(T)$ with $T := \{(1,1), \ldots, (t,1)\}$ is clearly a largest t-star of $\mathcal{L}_{\mathbf{k},r}$. $\square$

We mention that Erdős, Seress, and Székely [30] determined non-trivial t-intersecting sub-families of $\mathcal{L}_{\mathbf{k},r}$ of maximum size for the case when n is sufficiently large.

Finally, for the family $\mathcal{L}_{\mathbf{k},\leq n}$ of all labeled sets defined on the n-tuple $\mathbf{k}$, we have the following immediate consequence of Theorems 3.2 and 5.3.

Theorem 5.6. *For any $1 \leq k_1 \leq \cdots \leq k_n$, $\mathcal{L}_{(k_1,\ldots,k_n),\leq n}$ has the star property at $\{(1,1)\}$.*

Proof. Let $\mathbf{k} := (k_1, \ldots, k_n)$. If $k_1 = 1$ then $\mathcal{L}_{\mathbf{k},\leq n}$ is compressed with respect to $(1,1)$ and hence, since $\mathcal{L}_{\mathbf{k},\leq n}$ is hereditary, the result follows by Theorem 3.2. Now suppose $k_1 \geq 2$. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{L}_{\mathbf{k},\leq n}$. So $\emptyset \notin \mathcal{A}$. By Theorem 5.3, $|\mathcal{A}^{(r)}| \leq |\mathcal{L}_{\mathbf{k},r}((1,1))|$ for all $r \in [n]$. Thus, we have $|\mathcal{A}| = \sum_{r=1}^{n} |\mathcal{A}^{(r)}| \leq \sum_{r=1}^{n} |\mathcal{L}_{\mathbf{k},r}((1,1))| = |\mathcal{L}_{\mathbf{k},\leq n}((1,1))|$. $\square$

The above fact was also observed in [7], and it implies that the size of an intersecting sub-family of $\mathcal{L}_{\mathbf{k},\leq n}$ is at most $\frac{1}{k+1}|\mathcal{L}_{\mathbf{k},\leq n}|$, i.e. the size of the star $\mathcal{L}_{\mathbf{k},\leq n}((1,1))$ (indeed, the k_1+1 families $\mathcal{L}_{\mathbf{k},\leq n}((1,1)), \ldots, \mathcal{L}_{\mathbf{k},\leq n}((1,k_1))$ and $\mathcal{L}_{(k_2,\ldots,k_n),\leq n-1}$ partition $\mathcal{L}_{\mathbf{k},\leq n}$ and are of the same size). In view of the above-mentioned fact that $\mathcal{L}_{\mathbf{k},r}$ has the strict star property when $k_1 \geq 2$ and $(r,k_1) \neq (n,2)$ (in particular, when $1 \leq r \leq n-1$), one can go on to show that $\mathcal{L}_{\mathbf{k},\leq n}$ has the strict star property if $k_1 \geq 2$. If $k_1 = 1$ then $\mathcal{L}_{\mathbf{k},\leq n}$ may not have the strict star property; indeed, if $k_1 = k_2 = k_3 = 1$ then $\{A \in \mathcal{L}_{\mathbf{k},\leq n} \colon |A \cap \{(1,1),(2,1),(3,1)\}| \geq 2\}$ is a non-trivial intersecting sub-family that is as large as the largest star $\mathcal{L}_{\mathbf{k},\leq n}((1,1))$.

To the best of the author's knowledge, no general t-intersection theorem for $\mathcal{L}_{\mathbf{k},\leq n}$ is known.

6. Intersecting Families of Permutations and Partial Permutations

In [23, 24] the study of intersecting permutations was initiated. Deza and Frankl [24] showed that $\mathcal{S}^*_{[n],n}$ has the star property. So the size of an intersecting sub-family of $\mathcal{S}^*_{[n],n}$ is at most $(n-1)!$. The argument of the proof of this result is the same translation argument, given in the previous section, that yields Berge's intersection result for labeled sets [6], and it also gives us that for $n \leq k$, $\mathcal{S}^*_{[n],k}$ has the star property (recall that $\mathcal{S}^*_{[n],k} = \emptyset$ if $n > k$). Indeed, it gives us that for any intersecting sub-family $\mathcal{A}$ of $\mathcal{S}^*_{[n],k}$, $k|\mathcal{A}| \leq |\mathcal{S}^*_{[n],k}| = \frac{k!}{(k-n)!}$ and hence $|\mathcal{A}| \leq \frac{(k-1)!}{(k-n)!}$.

The question of whether $\mathcal{S}^*_{[n],n}$ has the strict star property proved to be much more difficult to answer. Cameron and Ku [18] and Larose and Malvenuto [47] independently gave an affirmative answer (other proofs are given in [35, 56]). Larose and Malvenuto [47] also proved the following generalisation (another proof is found in [17]).

Theorem 6.1 ([47]). *For $1 \leq n \leq k$, $\mathcal{S}^*_{[n],k}$ has the strict star property.*

Ku and Leader [46] investigated partial permutations. Using Katona's cycle method [40], they proved that $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the star property for all $r \in [n-1]$ (note that $\mathcal{S}^*_{\binom{[n]}{r},n} = \mathcal{S}^*_{[n],n}$ if $r = n$), and they also showed that $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the strict star property for all $r \in [8, n-3]$. Naturally, they conjectured that $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the strict star property for the few remaining values of r too. This was settled by Li and Wang [48] using tools forged by Ku and Leader. So the intersection results for $\mathcal{S}^*_{[n],n}$ and $\mathcal{S}^*_{\binom{[n]}{r},n}$ ($r \in [n-1]$) sum up as follows.

Theorem 6.2 ([18, 46–48]). *For any $r \in [n]$, $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the strict star property.*

When it comes to t-intersecting families of permutations, things are of course much harder. Solving a long-standing conjecture of Deza and Frankl [24], Ellis, Friedgut and Pilpel [26] recently managed to prove the following.

Theorem 6.3 ([26]). *For any integer $t \geq 1$, there exists an integer $n_0(t)$ such that for any $n \geq n_0(t)$, $\mathcal{S}^*_{[n],n}$ has the strict t-star property.*

Their remarkable proof is based on eigenvalue techniques and representation theory of the symmetric group. The condition $n \geq n_0(t)$ is necessary. Indeed, let $P_j := \{(i,i) \colon i \in [j]\}$ for any integer $j \geq 1$, and let

$$\mathcal{G}_{n,k,t} := \begin{cases} \{A \in \mathcal{S}_{[n],k} \colon |A \cap P_n| \geq (n+t)/2\} & \text{if } n-t \text{ is even;} \\ \{A \in \mathcal{S}_{[n],k} \colon |A \cap P_{n-1}| \geq (n+t-1)/2\} & \text{if } n-t \text{ is odd.} \end{cases}$$

Deza and Frankl [24] showed that when $t = n-s$ for some $s \geq 3$ and n is sufficiently large (depending on s), $\mathcal{G}_{n,n,t}$ is a largest t-intersecting sub-family of $\mathcal{S}^*_{[n],n}$ and is larger than the t-stars. Brunk and Huczynska [17] extended this result as follows.

Theorem 6.4 ([17, 24]). *For any integers $p \geq 0$ and $q \geq 2$ with $(p,q) \neq (0,2)$, there exists an integer $n^*_0(p,q)$ such that for any $n \geq n^*_0(p,q)$, any largest $(n-q)$-intersecting sub-family of $\mathcal{S}^*_{[n],n+p}$ is a copy of $\mathcal{G}_{n,n+p,n-q}$.*

They also conjectured that for any $n \leq k$ and $k \geq 8$, the extremal structures are similar to those in Theorem 2.2.

Conjecture 6.5 ([17]). *Let $1 \leq t \leq n \leq k$ and $k \geq 8$. Let $p := \lfloor (n-t)/2 \rfloor$, and for any integer i with $0 \leq i \leq p$, let $\mathcal{A}_i := \left\{A \in \mathcal{S}^*_{[n],k} \colon |A \cap P_{t+2i}| \geq t+i\right\}$. Then:*
(i) *one of the families $\mathcal{A}_0, \ldots, \mathcal{A}_p$ is a largest t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$;*
(ii) *any largest t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$ is a copy of one of the families $\mathcal{A}_0, \ldots, \mathcal{A}_p$.*

For the general case when $\mathcal{F}$ is any family, a conjecture for t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$ similar to Conjecture 4.8 was suggested in [15].

Conjecture 6.6 ([15]). *For any integer $t \geq 1$, there exists an integer $k_0^*(t)$ such that for any $k \geq k_0^*(t)$ and any family $\mathcal{F}$, $\mathcal{S}^*_{\mathcal{F},k}$ has the strict t-star property.*

Theorem 6.3 solves the special case $\mathcal{F} = \{[n]\}$ and $k = n \geq k_0^*(t)$. The author [15] proved the following relaxation of the statement of the conjecture.

Theorem 6.7 ([15]). *For any integers r and t with $1 \leq t < r$, let $k_0^*(r,t) := \binom{r}{t}\binom{3r-2t-1}{\lfloor \frac{3r-2t-1}{2} \rfloor}\frac{r!}{(r-t-1)!} + r + 1$. For any $k \geq k_0^*(r,t)$ and any family $\mathcal{F}$ with $\alpha(\mathcal{F}) \leq r$, $\mathcal{S}^*_{\mathcal{F},k}$ has the strict t-star property.*

This is an analogue of Theorem 4.9, and the general idea behind its proof is similar to that behind the proofs of Theorems 3.7 (see Section 3.) and 4.9.

By taking $\mathcal{F} = [n]$ and $k \geq k_0^*(n,t)$ in Theorem 6.7, we obtain the following.

Corollary 6.8. *Let $k \geq k_0^*(n,t)$, where $k_0^*(n,t)$ is as in Theorem 6.7. Then $\mathcal{S}^*_{[n],k}$ has the strict t-star property.*

Thus, when k is sufficiently large, the size of a t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$ is at most $\frac{(k-t)!}{(k-n)!}$.

The following t-intersection result for partial permutations is another immediate consequence of Theorem 6.7, obtained by taking $n \geq k_0^*(r,t)$ and $\mathcal{F} = \binom{[n]}{r}$.

Corollary 6.9. *Let $n \geq k_0^*(r,t)$, where $k_0^*(r,t)$ is as in Theorem 6.7. Then $\mathcal{S}^*_{\binom{[n]}{r},n}$ has the strict t-star property.*

Thus, when n is sufficiently large, the size of a t-intersecting sub-family of $\mathcal{S}^*_{\binom{[n]}{r},n}$ is at most $\binom{n-t}{r-t}\frac{(n-t)!}{(n-r)!}$. This was also proved in [45].

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