On t-intersecting families of signed sets and permutations

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Abstract

A family $\mathcal{A}$ of sets is said to be *t-intersecting* if any two sets in $\mathcal{A}$ contain at least t common elements. A t-intersecting family is said to be *trivial* if there are at least t elements common to all its sets.

Let X be an r-set $\{x_1, ..., x_r\}$. For $k \geq 2$, we define $\mathcal{S}_{X,k}$ and $\mathcal{S}^*_{X,k}$ to be the families of *k-signed r-sets* given by

$$\mathcal{S}_{X,k} := \{\{(x_1, a_1), ..., (x_r, a_r)\} \colon a_1, ..., a_r \text{ are elements of } \{1, ..., k\}\},$$
$$\mathcal{S}^*_{X,k} := \{\{(x_1, a_1), ..., (x_r, a_r)\} \colon a_1, ..., a_r \text{ are distinct elements of } \{1, ..., k\}\}.$$

$\mathcal{S}^*_{X,k}$ can be interpreted as the family of *permutations* of r-subsets of $\{1, ..., k\}$. For a family $\mathcal{F}$, we define $\mathcal{S}_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}$ and $\mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}^*_{F,k}$.

This paper features two theorems. The first one is as follows: For any two integers s and t with $t \leq s$, there exists an integer $k_0(s,t)$ such that, for any $k \geq k_0(s,t)$ and any family $\mathcal{F}$ with $t \leq \max\{|F| \colon F \in \mathcal{F}\} \leq s$, the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ are trivial. The second theorem is an analogue of the first one for $\mathcal{S}^*_{\mathcal{F},k}$.

1 Introduction

1.1 Notation and definitions

We start with some standard notation for sets. $\mathbb{N}$ is the set $\{1, 2, ...\}$ of positive integers. For $m, n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ is denoted by $[m, n]$, and if $m = 1$ then we also write $[n]$. For a set X, the *power set* $\{A \colon A \subseteq X\}$ of X is denoted by 2^X, and the *uniform* sub-family $\{Y \subseteq X \colon |Y| = r\}$ of 2^X is denoted by $\binom{X}{r}$.

For a family $\mathcal{F}$ of sets, we denote the union of all sets in $\mathcal{F}$ by $U(\mathcal{F})$. For a set V, we set

$$\mathcal{F}[V] := \{F \in \mathcal{F} \colon V \subseteq F\}, \quad \mathcal{F}(V) := \{F \in \mathcal{F} \colon F \cap V \neq \emptyset\}.$$

For $u \in U(\mathcal{F})$, we abbreviate $\mathcal{F}(\{u\})$ to $\mathcal{F}(u)$. We call $\mathcal{F}(u)$ a *star of $\mathcal{F}$*. More generally, if T is a t-subset of a set in $\mathcal{F}$, then we call $\mathcal{F}[T]$ a *t-star of $\mathcal{F}$*.

A family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$. More generally, $\mathcal{A}$ is said to be *t-intersecting* if $|A \cap B| \geq t$ for any $A, B \in \mathcal{A}$. A t-intersecting family $\mathcal{A}$ is said to be *trivial* if $|\bigcap_{A\in\mathcal{A}} A| \geq t$ (i.e. there are at least t elements common to all the sets in $\mathcal{A}$); otherwise, $\mathcal{A}$ is said to be *non-trivial.* Note that a t-star of a family $\mathcal{F}$ is a maximal trivial t-intersecting sub-family of $\mathcal{F}$.

In the following, unless otherwise stated, sets and families are to be assumed non-empty and finite.

1.2 Intersecting sub-families of $2^{[n]}$ and $\binom{[n]}{r}$

The study of intersecting families took off with the publication of [13], which features the classical result, known as the Erdős-Ko-Rado (EKR) Theorem, that says that, if $r \leq n/2$ and $\mathcal{A}$ is an intersecting sub-family of $\binom{[n]}{r}$, then $\mathcal{A}$ has size at most $\binom{n-1}{r-1}$, which is the size of a star of $\binom{[n]}{r}$. There are various proofs of this theorem, two of which are particularly short and beautiful: Katona's [21] using the *cycle method* and Daykin's [7] using another fundamental result known as the Kruskal-Katona Theorem [22, 25]. Hilton and Milner [19] determined the size of a largest non-trivial intersecting sub-family of $\binom{[n]}{r}$, and consequently they established that, if $r < n/2$, then no non-trivial intersecting sub-family of $\binom{[n]}{r}$ is as large as the stars of $\binom{[n]}{r}$.

The facts we have just mentioned inspire us to make the following definition. We say that a family $\mathcal{F}$ is *EKR* if the set of largest intersecting sub-families of $\mathcal{F}$ contains a star, and *strictly EKR* if the set of largest intersecting sub-families of $\mathcal{F}$ contains only stars.

Also in [13], Erdős, Ko and Rado initiated the study of t-intersecting families for $t \geq 2$. They pointed out the simple fact that $2^{[n]}$ is EKR, and they posed the following question: What is the size of an extremal (i.e. largest) t-intersecting sub-family of $2^{[n]}$ for $t \geq 2$? The answer in a complete form was given by Katona [23]. It is interesting that, for $n > t \geq 2$, no extremal t-intersecting sub-family of $2^{[n]}$ is a t-star.

For the uniform case, Erdős, Ko and Rado [13] proved that, for $t < r$, there exists an integer $n_0(r,t)$ such that, for all $n \geq n_0(r,t)$, the largest t-intersecting sub-families of $\binom{[n]}{r}$ are the t-stars. For $t \geq 15$, Frankl [14] showed that the smallest such $n_0(r,t)$ is $(r-t+1)(t+1)+1$ and that, if $n = (r-t+1)(t+1)$, then t-stars are extremal but not uniquely so. Subsequently, Wilson [33] proved the sharp upper bound $\binom{n-t}{r-t}$ for the size of a t-intersecting sub-family of $\binom{[n]}{r}$ for all t and $n \geq (r-t+1)(t+1)$. Frankl [14] conjectured that an extremal t-intersecting sub-family of $\binom{[n]}{r}$ has size $\max\{|\{A \in \binom{[n]}{r} \colon |A \cap [t+2i]| \geq t+i\}| \colon i \in \{0\} \cup [r-t]\}$. A remarkable proof of this long-standing conjecture together with a complete characterisation of the extremal structures was finally obtained by Ahlswede and Khachatrian [1].

Theorem 1.1 (Ahlswede and Khachatrian [1]) *Let $1 \leq t \leq r \leq n$, and let $\mathcal{A}$ be an extremal t-intersecting sub-family of $\binom{[n]}{r}$.*
(i) If $(r-t+1)(2+\frac{t-1}{i+1}) < n < (r-t+1)(2+\frac{t-1}{i})$ for some $i \in \{0\} \cup \mathbb{N}$ - where, by convention, $(t-1)/i = \infty$ if $i = 0$ - then $\mathcal{A} = \{A \in \binom{[n]}{r} \colon |A \cap X| \geq t+i\}$ for some $X \in \binom{[n]}{t+2i}$.
(ii) If $t \geq 2$ and $(r-t+1)(2+\frac{t-1}{i+1}) = n$ for some $i \in \{0\} \cup \mathbb{N}$, then $\mathcal{A} = \{A \in \binom{[n]}{r} \colon |A \cap X| \geq t+j\}$ for some $j \in \{i, i+1\}$ and $X \in \binom{[n]}{t+2j}$.

Many other beautiful results were inspired by the seminal Erdős-Ko-Rado paper [13]. The survey papers [10] and [15] are recommended.

We now proceed to the first of the two main themes of the paper.

1.3 Intersecting families of signed sets

Let X be an r-set $\{x_1, ..., x_r\}$. Let $y_1, ..., y_r \in \mathbb{N}$. We call the set $\{(x_1, y_1), ..., (x_r, y_r)\}$ a *k-signed r-set* if $|\{y_1, ..., y_r\}| \leq k$. For an integer $k \geq 2$, we define $\mathcal{S}_{X,k}$ to be the family of k-signed r-sets given by

$$\mathcal{S}_{X,k} := \{\{(x_1, a_1), ..., (x_r, a_r)\} \colon a_1, ..., a_r \in [k]\}.$$

We shall set $\mathcal{S}_{\emptyset,k} := \emptyset$.

The *Cartesian product* $X \times Y$ of sets X and Y is the set $\{(x, y) \colon x \in X, y \in Y\}$. So $\mathcal{S}_{X,k} = \{A \subset X \times [k] \colon |A \cap (\{x\} \times [k])| = 1 \text{ for all } x \in X\}$.

For a family $\mathcal{F}$ of sets, we define

$$\mathcal{S}_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}.$$

We remark that the 'signed sets' terminology was introduced in [4] for a setting that can be re-formulated as $\mathcal{S}_{\binom{[n]}{r},k}$, and the general formulation $\mathcal{S}_{\mathcal{F},k}$ was introduced by the author in [5], the theme of which is the following conjecture.

Conjecture 1.2 (Borg [5]) *Let $\mathcal{F}$ be any family, and let $k \geq 2$. Then:*
(i) $\mathcal{S}_{\mathcal{F},k}$ is EKR;
(ii) $\mathcal{S}_{\mathcal{F},k}$ is not strictly EKR iff $k = 2$ and there exist at least three elements u_1, u_2, u_3 of $U(\mathcal{F})$ such that $\mathcal{F}(u_1) = \mathcal{F}(u_2) = \mathcal{F}(u_3)$ and $\mathcal{S}_{\mathcal{F},2}((u_1, 1))$ is a largest star of $\mathcal{S}_{\mathcal{F},2}$.

The main result in the same paper is that this conjecture is true if $\mathcal{F}$ is compressed with respect to an element u^* of $U(\mathcal{F})$ (i.e. $u \in F \in \mathcal{F} \backslash \mathcal{F}(u^*)$ implies $(F \backslash \{u\}) \cup \{u^*\} \in \mathcal{F}$). This generalises a well-known result that was first stated by Meyer [31] and proved in different ways by Deza and Frankl [10], Bollobás and Leader [4], Engel [11] and Erdős et al. [12], and that can be described as saying that the conjecture is true for $\mathcal{F} = \binom{[n]}{r}$. Berge [3] and Livingston [30] had proved (i) and (ii) respectively for the special case $\mathcal{F} = \{[n]\}$ (other proofs are found in [18, 32]). In [5] the conjecture is also verified for $\mathcal{F}$ uniform and EKR; Holroyd and Talbot [20] had essentially proved (i) for such a family $\mathcal{F}$ in a graph-theoretical context.

The t-intersection problem for sub-families of $\mathcal{S}_{[n],k}$ has also been solved. Frankl and Füredi [16] were the first to investigate it, and the following result had been a conjecture that they made and that they verified for $k \geq t + 1 \geq 16$ in [16].

Theorem 1.3 (Ahlswede, Khachatrian [2]; Frankl, Tokushige [17]) *If $\mathcal{A}$ is an extremal t-intersecting sub-family of $\mathcal{S}_{[n],k}$, then $|\mathcal{A}| = \max\{|\{A \in \mathcal{S}_{[n],k} \colon |A \cap ([t + 2i] \times [1])| \geq t + i\}| \colon i \in \{0\} \cup \mathbb{N}\}$.*

It follows from this result that the set of extremal t-intersecting sub-families of $\mathcal{S}_{[n],k}$ contains t-stars iff $k \geq t + 1$. What led to this result was the accomplishment of Theorem 1.1. As in Theorem 1.1, Ahlswede and Khachatrian [2] also determined the extremal t-intersecting sub-families of $\mathcal{S}_{[n],k}$, and it turns out that the structure of a t-star of $\mathcal{S}_{[n],k}$ is the unique extremal structure iff $k \geq t + 2$. Kleitman [24] had long established Theorem 1.3 for $k = 2$.

To the best of the author's knowledge, apart from a general result we present later, no results for t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ with $|\mathcal{F}| \geq 2$ have been established. However, some very important results have been obtained for a modification of the problem, which we describe next.

1.4 Intersecting families of permutations and partial permutations

For an r-set $X := \{x_1, ..., x_r\}$, we define $\mathcal{S}^*_{X,k}$ to be the special sub-family of $\mathcal{S}_{X,k}$ given by

$$\mathcal{S}^*_{X,k} := \left\{\{(x_1, a_1), ..., (x_r, a_r)\} \colon \{a_1, ..., a_r\} \in \binom{[k]}{r}\right\}.$$

Note that $\mathcal{S}^*_{X,k} \neq \emptyset$ iff $r \leq k$.

For a family $\mathcal{F}$, we define $\mathcal{S}^*_{\mathcal{F},k}$ to be the special sub-family of $\mathcal{S}_{\mathcal{F},k}$ given by

$$\mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}^*_{F,k}.$$

An *r-partial permutation of a set* N is a pair (A, f) where $A \in \binom{N}{r}$ and $f \colon A \to N$ is an injection. An $|N|$-partial permutation of N is simply called a *permutation of* N. Clearly, the family of permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{[n],n}$, and the family of r-partial permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{\binom{[n]}{r},n}$.

Let X be as above. $\mathcal{S}^*_{X,k}$ can be interpreted as the family of permutations of sets in $\binom{[k]}{r}$: consider the bijection $\beta \colon \mathcal{S}^*_{X,k} \to \{(A, f) \colon A \in \binom{[k]}{r}, f \colon A \to A \text{ is a bijection}\}$ defined by $\beta(\{(x_1, a_1), ..., (x_r, a_r)\}) := (\{a_1, ..., a_r\}, f)$ where, for $b_1 < ... < b_r$ such that $\{b_1, ..., b_r\} = \{a_1, ..., a_r\}$, $f(b_i) := a_i$ for $i = 1, ..., r$. $\mathcal{S}^*_{X,k}$ can also be interpreted as the sub-family $\mathcal{X} := \{(A, f) \colon A \in \binom{[k]}{r}, f \colon A \to [r] \text{ is a bijection}\}$ of the family of r-partial permutations of $[k]$: consider an obvious bijection from $\mathcal{S}^*_{X,k}$ to $\mathcal{S}^*_{\binom{[k]}{r},r}$ and another one from $\mathcal{S}^*_{\binom{[k]}{r},r}$ to $\mathcal{X}$.

In [8, 9] the study of intersecting permutations was initiated. Deza and Frankl [9] showed that $\mathcal{S}^*_{[n],n}$ is EKR. So an intersecting sub-family of $\mathcal{S}^*_{[n],n}$ has size at most $(n-1)!$. Only a few years ago, Cameron and Ku [6] and Larose and Malvenuto [28] independently proved that furthermore $\mathcal{S}^*_{[n],n}$ is strictly EKR.

Ku and Leader [27] proved that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is EKR for all $r \in [n]$, and they also showed that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is strictly EKR for all $r \in [8, n-3]$. Naturally, they conjectured that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is also strictly EKR for the few remaining values of r. This was settled by Li and Wang [29] using tools forged by Ku and Leader.

When it comes to t-intersecting families of permutations, things are of course much harder, and the most interesting challenge comes from the following conjecture.

Conjecture 1.4 (Deza and Frankl [9]) *For any $t \in \mathbb{N}$, there exists $n_0(t) \in \mathbb{N}$ such that, for any $n \geq n_0(t)$, the size of a t-intersecting sub-family of $\mathcal{S}^*_{[n],n}$ is at most that of a t-star of $\mathcal{S}^*_{[n],n}$, i.e. $(n-t)!$.*

This conjecture suggests an obvious extension for the extremal case. It is worth pointing out that the condition $n \geq n_0(t)$ is necessary; [26, Example 3.1.1] illustrates this fact. An analogue of the statement of the conjecture for partial permutations has been proved by Ku.

Theorem 1.5 (Ku [26, Theorem 6.6.6]) *For any $r, t \in \mathbb{N}$ with $r \geq t$, there exists $n_0(r, t) \in \mathbb{N}$ such that, for any $n \geq n_0(r, t)$, the size of a t-intersecting sub-family of $\mathcal{S}^*_{\binom{[n]}{r},n}$ is at most that of a t-star of $\mathcal{S}^*_{\binom{[n]}{r},n}$, i.e. $\binom{n-t}{r-t}\frac{(n-t)!}{(n-r)!}$.*

This result emerges as an immediate consequence of one of the two main theorems in this paper; see next section.

2 Results and conjectures

For a family $\mathcal{F}$, let $\alpha(\mathcal{F})$ denote the size of a largest set in $\mathcal{F}$. Any t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$ or $\mathcal{S}^*_{\mathcal{F},k}$ trivially consists of at most one set if $\alpha(\mathcal{F}) \leq t$. We now consider $\alpha(\mathcal{F}) > t$.

In view of Conjecture 1.2, we suggest the following general conjecture for t-intersecting families of signed sets.

Conjecture 2.1 *For any $t \in \mathbb{N}$, there exists $k_0(t) \in \mathbb{N}$ such that, for any $k \geq k_0(t)$ and any family $\mathcal{F}$ with $\alpha(\mathcal{F}) > t$, the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ are trivial.*

As we mentioned in Section 1.3, the t-stars of $\mathcal{S}_{[n],k}$ are extremal t-intersecting sub-families of $\mathcal{S}_{[n],k}$ iff $k \geq t+1$, and they are uniquely extremal iff $k \geq t+2$. This suggests that, if Conjecture 2.1 is true, then, as is claimed by Conjecture 1.2 for $t = 1$, the smallest value of $k_0(t)$ is $t+2$ (and the largest t-stars of $\mathcal{S}_{\mathcal{F},t+1}$ are among the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},t+1}$). We are able to prove a relaxation of the statement of Conjecture 2.1.

Theorem 2.2 *For any $r, t \in \mathbb{N}$ with $t < r$, let $k_0(r,t) := \binom{r}{t}\binom{r}{t+1}$. For any $k \geq k_0(r,t)$ and any family $\mathcal{F}$ with $t < \alpha(\mathcal{F}) \leq r$, the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ are trivial.*

Corollary 2.3 *Conjecture 1.2 is true if $k \geq \alpha(\mathcal{F})\binom{\alpha(\mathcal{F})}{2}$.*

We next pose a similar problem for t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$.

Conjecture 2.4 *For any $t \in \mathbb{N}$, there exists $k^*_0(t) \in \mathbb{N}$ such that, for any $k \geq k^*_0(t)$ and any family $\mathcal{F}$ with $\alpha(\mathcal{F}) > t$, the largest t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$ are trivial.*

By taking $k \geq k^*_0(t)$ and $\mathcal{F} = \{[k]\}$, we get Conjecture 1.4. We are able to prove the following analogue of Theorem 2.2.

Theorem 2.5 *For any $r, t \in \mathbb{N}$ with $t < r$, let $k^*_0(r,t) := \binom{r}{t}\binom{3r-2t-1}{\lfloor \frac{3r-2t-1}{2} \rfloor}\frac{r!}{(r-t-1)!} + r + 1$. For any $k \geq k^*_0(r,t)$ and any family $\mathcal{F}$ with $t < \alpha(\mathcal{F}) \leq r$, the largest t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$ are trivial.*

By taking $k \geq k^*_0(r,t)$ and $\mathcal{F} = \binom{[k]}{r}$, we get Theorem 1.5.

We now proceed to the proofs of the two theorems above.

3 Proof of Theorem 2.2

We shall base the proof of Theorem 2.2 on the compression technique used in [10] and in [16]. We point out that this can be avoided by applying an argument similar to the one for Theorem 2.5; however, the compression technique enables us to obtain a neater proof and a value of $k_0(r,t)$ that is better than what we would obtain without using it.

For $(a,b) \in [n] \times [2,k]$, let $\Delta_{a,b} \colon 2^{\mathcal{S}_{2^{[n]},k}} \to 2^{\mathcal{S}_{2^{[n]},k}}$ be defined by

$$\Delta_{a,b}(\mathcal{A}) := \{\delta_{a,b}(A) \colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \delta_{a,b}(A) \in \mathcal{A}\},$$

where $\delta_{a,b} \colon \mathcal{S}_{2^{[n]},k} \to \mathcal{S}_{2^{[n]},k}$ is defined by

$$\delta_{a,b}(A) := \begin{cases} A \backslash \{(a,b)\} \cup \{(a,1)\} & \text{if } (a,b) \in A; \\ A & \text{otherwise} \end{cases}$$

Note that $|\Delta_{a,b}(\mathcal{A})| = |\mathcal{A}|$. It is known and easy to check that, if $\mathcal{A}$ is t-intersecting, then $\Delta_{a,b}(\mathcal{A})$ is t-intersecting. We prove a bit more than this.

Lemma 3.1 *Let $\mathcal{A} \subset \mathcal{S}_{2^{[n]},k}$ and $V \subseteq [n] \times [2,k]$ such that $|(A \cap B) \backslash V| \geq t$ for any $A, B \in \mathcal{A}$. Then $|(C \cap D) \backslash (V \cup \{(a,b)\})| \geq t$ for any $C, D \in \Delta_{a,b}(\mathcal{A})$.*

Proof. Let $C, D \in \Delta_{a,b}(\mathcal{A})$. Let $C' := (C \backslash \{(a,1)\}) \cup \{(a,b)\}$, $D' := (D \backslash \{(a,1)\}) \cup \{(a,b)\}$. Suppose $|(C \cap D) \backslash V| < t$. So C and D cannot both be in $\mathcal{A}$. Suppose $C, D \notin \mathcal{A}$; then $(a,1)$ is in both C and D, C' and D' are in $\mathcal{A}$, and $|(C' \cap D') \backslash V| \leq |(C \cap D) \backslash V| < t$, a contradiction. Thus, without loss of generality, $C \notin \mathcal{A}$ and $D \in \mathcal{A}$. So $(a,1) \in C$ and $C' \in \mathcal{A}$. If $(a,b) \notin D$ then $|(C' \cap D) \backslash V| \leq |(C \cap D) \backslash V| < t$, contradicting $C', D \in \mathcal{A}$. So $(a,b) \in D$ and hence $\delta_{a,b}(D) \in \mathcal{A}$ (because otherwise $D \notin \Delta_{a,b}(\mathcal{A})$). But then $|(C' \cap \delta_{a,b}(D)) \backslash V| = |(C \cap D) \backslash V| < t$, contradicting $C', \delta_{a,b}(D) \in \mathcal{A}$. We therefore conclude that $|(C \cap D) \backslash V| \geq t$.

Now suppose $|(C \cap D) \backslash (V \cup \{(a,b)\})| < t$. Since $|(C \cap D) \backslash V| \geq t$, $(a,b) \in C \cap D$. So $C, \delta_{a,b}(C), D, \delta_{a,b}(D) \in \mathcal{A}$ and $|(C \cap \delta_{a,b}(D)) \backslash V| = |(C \cap D) \backslash (V \cup \{(a,b)\})| < t$, a contradiction. $\square$

Corollary 3.2 *Let $\mathcal{A}^*$ be a t-intersecting sub-family of $\mathcal{S}_{2^{[n]},k}$. Let*

$$\mathcal{A} := \Delta_{n,k} \circ ... \circ \Delta_{n,2} \circ ... \circ \Delta_{1,k} \circ ... \circ \Delta_{1,2}(\mathcal{A}^*).$$

Then $|A \cap B \cap ([n] \times [1])| \geq t$ for any $A, B \in \mathcal{A}$.

Proof. By repeated application of Lemma 3.1, $|(A \cap B) \backslash ([n] \times [2,k])| \geq t$ for any $A, B \in \mathcal{A}$. The result follows since $(A \cap B) \backslash ([n] \times [2,k]) = A \cap B \cap ([n] \times [1])$. $\square$

Lemma 3.3 *Let $\mathcal{F} \subseteq 2^{[n]}$, $k \geq 3$ and $(a,b) \in [n] \times [2,k]$. Suppose $\mathcal{A}$ is a non-trivial t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$ and $\Delta_{a,b}(\mathcal{A})$ is a sub-family of a t-star $\mathcal{S}_{\mathcal{F},k}[Z]$ ($Z \in \mathcal{S}_{\binom{[n]}{t},k}$) of $\mathcal{S}_{\mathcal{F},k}$. Then $|\mathcal{A}| < |\mathcal{S}_{\mathcal{F},k}[Z]|$.*

Proof. Let $Y := \{z \colon (z,l) \in Z \text{ for some } l \in [k]\}$. Given that $\Delta_{a,b}(\mathcal{A}) \subseteq \mathcal{S}_{\mathcal{F},k}[Z]$, we have $\mathcal{A} \subset \mathcal{S}_{\mathcal{F}[Y],k}$ and, since $\mathcal{A}$ is non-trivial, there exists $A \in \mathcal{A}$ such that $|A \cap Z| = t-1$ and $Z \subseteq \delta_{a,b}(A)$. So $(a,1) \in Z$ and $Z' := Z \backslash \{(a,1)\} \subset A$ for all $A \in \mathcal{A}$. Let $Y' := Y \backslash \{a\}$. Setting $\mathcal{F}' := \{F \backslash Y' \colon F \in \mathcal{F}[Y']\}$ and $\mathcal{A}' := \{A \backslash Z' \colon A \in \mathcal{A}[Z']\}$, we then have $\mathcal{A}' \subset \mathcal{S}_{\mathcal{F}'(a),k}$ (as $\mathcal{A} \subset \mathcal{S}_{\mathcal{F}[Y],k}$ and $Y = Y' \cup \{a\}$) and $|\mathcal{A}'| = |\mathcal{A}|$. Since $\mathcal{A}$ is a non-trivial t-intersecting family and $|Z'| = t-1$, $\mathcal{A}'$ is a non-trivial intersecting family.

For $F' \in \mathcal{F}'(a)$, let $\mathcal{A}'_{F'} := \mathcal{A}' \cap \mathcal{S}_{F',k}$. Since $\mathcal{A}'$ is intersecting, $\mathcal{A}'_{F'}$ is intersecting. Suppose $\mathcal{A}'_{F'} \neq \emptyset$. If $\mathcal{A}'_{F'}$ is non-trivial, then, by Livingston's theorem [30] (see Section 1.3), $|\mathcal{A}'_{F'}| < k^{|F'|-1}$. Suppose $\mathcal{A}'_{F'}$ is trivial; so $\mathcal{A}'_{F'} \subseteq \mathcal{S}_{F',k}((c,d))$ for some $(c,d) \in F' \times [k]$. Since $\mathcal{A}'$ is non-trivial, there exists $A' \in \mathcal{A}'$ such that $(c,d) \notin A'$. Thus, since $\mathcal{A}'$ is intersecting, we actually have $\mathcal{A}'_{F'} \subseteq \{A \in \mathcal{S}_{F',k}((c,d)) \colon A \cap A' \neq \emptyset\}$, and hence we again get $|\mathcal{A}'_{F'}| < k^{|F'|-1}$.

We therefore have

$$|\mathcal{A}| = |\mathcal{A}'| = \sum_{F' \in \mathcal{F}'(a)} |\mathcal{A}'_{F'}| < \sum_{F' \in \mathcal{F}'(a)} k^{|F'|-1} = \sum_{F \in \mathcal{F}[Y]} k^{|F|-t},$$

and the result follows since $\sum_{F \in \mathcal{F}[Y]} k^{|F|-t} = |\mathcal{S}_{\mathcal{F},k}[Z]|$. $\square$

Proof of Theorem 2.2. Let $\mathcal{F}$ be a family with $t < \alpha(\mathcal{F}) \leq r$. We may assume that $\mathcal{F} \subseteq 2^{[n]}$ for some $n \in \mathbb{N}$. Let $k \geq k_0(r,t)$. We prove the result by showing that, for any non-trivial t-intersecting sub-family $\mathcal{B}$ of $\mathcal{S}_{\mathcal{F},k}$, there exists a trivial t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$ that is larger than $\mathcal{B}$.

Let $\mathcal{A}^*$ be a non-trivial t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$. Let $\mathcal{A} := \Delta_{n,k} \circ ... \circ \Delta_{n,2} \circ ... \circ \Delta_{1,k} \circ ... \circ \Delta_{1,2}(\mathcal{A}^*)$. So $\mathcal{A} \subset \mathcal{S}_{\mathcal{F},k}$ and $|\mathcal{A}| = |\mathcal{A}^*|$. Let $X := [n] \times [1]$. By Corollary 3.2,

$$|A \cap B \cap X| \geq t \text{ for any } A, B \in \mathcal{A}. \tag{1}$$

Suppose $\mathcal{A}$ is a trivial t-intersecting family, i.e. $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}[Z]$ for some $Z \in \binom{S}{t}$, $S \in \mathcal{S}_{\mathcal{F},k}$. By Lemma 3.3, we then have $|\mathcal{A}^*| < |\mathcal{S}_{\mathcal{F},k}[Z]|$, and hence we are done.

We now assume $\mathcal{A}$ is a non-trivial t-intersecting family. Suppose $|A' \cap X| = t$ for some $A' \in \mathcal{A}$. Then, by (1), $A' \cap X \subseteq A$ for all $A \in \mathcal{A}$; but this contradicts the assumption that $\mathcal{A}$ is non-trivial. So $|A \cap X| \geq t+1$ for all $A \in \mathcal{A}$, and hence we obtain a crude bound for the size of $\mathcal{A}_F := \mathcal{A} \cap \mathcal{S}_{F,k}$ ($F \in \mathcal{F}$) as follows:

$$|\mathcal{A}_F| \leq |\{A \in \mathcal{S}_{F,k} \colon |A \cap (F \times [1])| \geq t+1\}| < \binom{|F|}{t+1} k^{|F|-t-1} \leq \binom{r}{t+1} k^{|F|-t-1}. \tag{2}$$

Let $B \in \mathcal{A}$. Since $\mathcal{A}$ is t-intersecting (by (1)), each $A \in \mathcal{A}$ must contain at least one of the sets in $\binom{B}{t}$, and hence $\mathcal{A} = \bigcup_{C \in \binom{B}{t}} \mathcal{A}[C]$. Choose $C^* \in \binom{B}{t}$ such that $|\mathcal{A}[C]| \leq |\mathcal{A}[C^*]|$ for all $C \in \binom{B}{t}$. We then have

$$|\mathcal{A}| = |\bigcup_{C \in \binom{B}{t}} \mathcal{A}[C]| \leq \sum_{C \in \binom{B}{t}} |\mathcal{A}[C]| \leq \binom{|B|}{t} |\mathcal{A}[C^*]| \leq \binom{r}{t} |\mathcal{A}[C^*]|. \tag{3}$$

Set $\mathcal{G} := \{F \in \mathcal{F} \colon \mathcal{A}[C^*] \cap \mathcal{S}_{F,k} \neq \emptyset\}$. Let $\mathcal{C}$ be the trivial t-intersecting sub-family $\bigcup_{G \in \mathcal{G}} \mathcal{S}_{G,k}[C^*]$ of $\mathcal{S}_{\mathcal{F},k}$. Bringing all the pieces together, we get

$$\begin{aligned}
|\mathcal{A}| &\leq \binom{r}{t} |\mathcal{A}[C^*]| && \text{(by (3))} \\
&\leq \binom{r}{t} \sum_{G \in \mathcal{G}} |\mathcal{A}_G| = \sum_{G \in \mathcal{G}} \binom{r}{t} |\mathcal{A}_G| \\
&< \sum_{G \in \mathcal{G}} \binom{r}{t} \binom{r}{t+1} k^{|G|-t-1} && \text{(by (2))} \\
&= \sum_{G \in \mathcal{G}} k_0(r,t) k^{|G|-t-1} \leq \sum_{G \in \mathcal{G}} k^{|G|-t} = |\mathcal{C}|.
\end{aligned}$$

So $|\mathcal{A}^*| < |\mathcal{C}|$ as $|\mathcal{A}^*| = |\mathcal{A}|$. Hence the result. □

4 Proof of Theorem 2.5

The proof of Theorem 2.5 is based on ideas from the preceding section and ideas used by Erdős, Ko and Rado [13] for their result concerning t-intersecting sub-families of $\binom{[n]}{r}$. Unfortunately, the compression technique fails to work for intersecting sub-families of $\mathcal{S}^*_{[n],k}$.

Let $l(n,k,t)$ be the size of a largest non-trivial t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$, and let $P_j := \{(i,i) \colon i \in [j]\}$.

Lemma 4.1 *For any $c, n, t \in \mathbb{N}$ with $t < n$, let $k_0(c,n,t) := c\binom{3n-2t-1}{\lfloor \frac{3n-2t-1}{2} \rfloor} \frac{n!}{(n-t-1)!} + n + 1$. For any $k \geq k_0(c,n,t)$,*

$$|\mathcal{S}^*_{[n],k}[P_t]| > c(\max\{l(n,k,t), |\mathcal{S}^*_{[n],k}[P_{t+1}]|\}).$$

Proof. Let $k \geq k_0(c,n,t)$, and let $\mathcal{A} \subset \mathcal{S}^*_{[n],k}$ be a non-trivial t-intersecting family of size $l(n,k,t)$. Choose $A_1, A_2 \in \mathcal{A}$ such that $|A_1 \cap A_2| \leq |A \cap B|$ for all $A, B \in \mathcal{A}$.

Suppose $|A_1 \cap A_2| \geq t+1$. Let $(i^*, j^*) \in [n] \times [k]$ such that $(i^*, j^*) \in A_1 \cap A_2$. Let $j' \in [k]$ such that $(i, j') \notin A_1 \cup A_2$ for all $i \in [n]$ (note that such a j' exists since $k \geq k_0(c,n,t) > |A_1 \cup A_2|$). Let $A_1' := (A_1 \backslash \{(i^*, j^*)\}) \cup (i^*, j')$. By choice of j', $A_1' \in \mathcal{S}^*_{[n],k}$. Let $\mathcal{A}' := \mathcal{A} \cup \{A_1'\}$. Since $|A_1' \cap A_2| < |A_1 \cap A_2|$, it follows by choice of A_1 and A_2 that $A_1' \notin \mathcal{A}$ and hence $|\mathcal{A}'| = |\mathcal{A}| + 1$. Also by choice of A_1 and A_2, we have $|A \cap B| \geq t+1$ for all $A, B \in \mathcal{A}$, which implies that $\mathcal{A}'$ is t-intersecting. Since $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{A}$ is non-trivially t-intersecting, $|\bigcap_{A' \in \mathcal{A}'} A'| \leq |\bigcap_{A \in \mathcal{A}} A| < t$. So $\mathcal{A}'$ is a non-trivial t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$ of size greater than $|\mathcal{A}|$; but this contradicts $|\mathcal{A}| = l(n,k,t)$. We therefore conclude that $|A_1 \cap A_2| = t$. Thus, since $\mathcal{A}$ is non-trivially t-intersecting, there exists $A_3 \in \mathcal{A}$ such that $A_1 \cap A_2 \nsubseteq A_3$ and hence $|A_1 \cap A_2 \cap A_3| < t$.

Let $I := A_1 \cup A_2 \cup A_3$. Suppose there exists $A^* \in \mathcal{A}$ such that $|A^* \cap I| < t+1$. Since $|A_1 \cap A_2| = t$ and $|A^* \cap A_i| \geq t$ for each $i \in [2]$, we must then have $A^* \cap (A_1 \cup A_2) = A_1 \cap A_2$. Thus, by our supposition, $A^* \cap I = A_1 \cap A_2$. But then $A^* \cap A_3 = A_1 \cap A_2 \cap A_3$, which gives the contradiction that $|A^* \cap A_3| < t$. Therefore

$$|A \cap I| \geq t+1 \text{ for all } A \in \mathcal{A}. \tag{4}$$

Now $|I| = |A_1 \cup A_2| + |A_3| - |A_3 \cap (A_1 \cup A_2)|$. Since $|A_1 \cup A_2| = 2n - |A_1 \cap A_2| = 2n - t$ and $|A_3 \cap (A_1 \cup A_2)| = |A_3 \cap A_1| + |(A_3 \cap A_2) \backslash A_1| \geq t + (t - |A_3 \cap A_2 \cap A_1|) \geq 2t - (t-1) = t+1$, it follows that

$$|I| \leq (2n-t) + n - (t+1) = 3n - 2t - 1.$$

Taking J to be the smallest set such that $I \subset [n] \times J$, we then have

$$n \leq |J| \leq 3n - 2t - 1.$$

For each $i \in [t+1, n]$, let $\mathcal{A}_i := \{A \in \mathcal{A} \colon |A \cap ([n] \times J)| = i\}$. By (4), $\bigcup_{i=t+1}^n \mathcal{A}_i$ is a partition for $\mathcal{A}$. Let $x := \sum_{i=t+1}^n |\{A \in \mathcal{S}^*_{[n],k} \colon |A \cap ([n] \times J)| = i\}|$. We therefore have

$$\begin{aligned}
l(n,k,t) = |\mathcal{A}| = \sum_{i=t+1}^{n} |\mathcal{A}_i| < x &= \sum_{i=t+1}^{n} \binom{|J|}{i} \binom{n}{i} i! \binom{k-|J|}{n-i} (n-i)! \\
&< \sum_{i=t+1}^{n} \binom{3n-2t-1}{i} \binom{n}{i} i! \binom{k-n}{n-i} (n-i)! \\
&\leq \sum_{i=t+1}^{n} \binom{3n-2t-1}{i} \frac{n!}{(n-i)!} (k-n)^{(n-i)} \\
&\leq \binom{3n-2t-1}{\lfloor \frac{3n-2t-1}{2} \rfloor} \frac{n!}{(n-t-1)!} \sum_{i=t+1}^{n} (k-n)^{(n-i)} \\
&= \left(\frac{k_0(c,n,t) - n - 1}{c} \right) \left(\frac{1 - (k-n)^{n-t}}{1 - (k-n)} \right) \leq \frac{(k-n)^{n-t} - 1}{c} \\
&< \frac{1}{c} \left(\frac{(k-t)!}{(k-n)!} \right) = \frac{|\mathcal{S}^*_{[n],k}[P_t]|}{c}.
\end{aligned}$$

The result now follows since we also have $|\mathcal{S}^*_{[n],k}[P_{t+1}]| < x$. $\square$

Proof of Theorem 2.5. Let $\mathcal{F}$ be a family with $t < \alpha(\mathcal{F}) \leq r$. Let $k_0(\binom{r}{t}, n, t)$ be as in the statement of Lemma 4.1 with $c = \binom{r}{t}$. Let $k \geq k_0^*(r,t)$. So we have

$$k \geq k_0(\binom{r}{t}, r, t) = \max\{k_0(\binom{r}{t}, n, t) \colon n \in [r]\}. \tag{5}$$

Let $\mathcal{A}$ be a non-trivial t-intersecting sub-family of $\mathcal{S}^*_{\mathcal{F},k}$.

For any $F \in \mathcal{F}$ and any family $\mathcal{B} \subseteq \mathcal{S}^*_{\mathcal{F},k}$, set $\mathcal{B}_F := \mathcal{B} \cap \mathcal{S}^*_{F,k}$. For all $F \in \mathcal{F}$, choose $F' \in \mathcal{S}^*_{\binom{F}{t},k}$. We show that, for all $F \in \mathcal{F}$,

$$\binom{r}{t}|\mathcal{A}_F| < |\mathcal{S}^*_{F,k}[F']|. \tag{6}$$

If $\mathcal{A}_F$ is a non-trivial t-intersecting family, then (6) follows immediately from (5) and Lemma 4.1. Now suppose $\mathcal{A}_F$ is a trivial t-intersecting family. Setting $T := \bigcap_{A \in \mathcal{A}_F} A$, we then have $|T| \geq t$. If $|T| \geq t+1$, then (6) again follows immediately from (5) and Lemma 4.1. It remains to consider $|T| = t$. Since $\mathcal{A}$ is a non-trivial t-intersecting family, there exists $A_1 \in \mathcal{A}$ such that $T \nsubseteq A_1$ and hence $|T \cap A_1| < t$. Let $D_1 := A_1 \cap (F \times [k])$. Let F_1 be the subset of F such that $D_1 \in \mathcal{S}^*_{F_1,k}$. Let $F_2 := F \backslash F_1$. Let $Y := \{y \in [k] \colon (x,y) \notin D_1 \cup T \text{ for all } x \in F\}$, and let $y_1, ..., y_{|Y|}$ be the distinct elements of Y. We have $|Y| \geq k - |D_1| - |T| = k - |F_1| - t = k - (|F| - |F_2|) - t \geq k_0^*(r,t) - r - t + |F_2| > |F_2|$. If $F_2 \neq \emptyset$ and $x_1, ..., x_{|F_2|}$ are the distinct elements of F_2, then we take D_2 to be the set $\{(x_1, y_1), ..., (x_{|F_2|}, y_{|F_2|})\}$ in $\mathcal{S}^*_{F_2,k}$; otherwise we take $D_2 := \emptyset$. Let $A_2 := D_1 \cup D_2$. Clearly $A_2 \in \mathcal{S}^*_{F,k}$. Therefore $\mathcal{A}_F \cup \{A_2\}$ is a non-trivial t-intersecting sub-family of $\mathcal{S}^*_{F,k}$ because $|\bigcap_{A' \in \mathcal{A}_F \cup \{A_2\}} A'| = |T \cap A_2| = |T \cap D_1| = |T \cap A_1| < t$ and, for all $A \in \mathcal{A}_F$, $|A_2 \cap A| \geq |D_1 \cap A| = |A_1 \cap A| \geq t$. By (5) and Lemma 4.1, it follows that $\binom{r}{t}|\mathcal{A}_F \cup \{A_2\}| < |\mathcal{S}^*_{F,k}[F']|$, and hence (6).

Now, as in the proof of Theorem 2.2, by choosing $B \in \mathcal{A}$ and $C^* \in \binom{B}{t}$ such that $|\mathcal{A}[C]| \leq |\mathcal{A}[C^*]|$ for all $C \in \binom{B}{t}$, we get

$$|\mathcal{A}| \leq \binom{r}{t}|\mathcal{A}[C^*]|.$$

Set $\mathcal{G} := \{F \in \mathcal{F} \colon \mathcal{A}[C^*] \cap \mathcal{S}^*_{F,k} \neq \emptyset\}$. Let $\mathcal{C}$ be the trivial t-intersecting sub-family $\bigcup_{G \in \mathcal{G}} \mathcal{S}^*_{G,k}[C^*]$ of $\mathcal{S}^*_{\mathcal{F},k}$. Bringing all the pieces together, we get

$$|\mathcal{A}| \leq \binom{r}{t}|\mathcal{A}[C^*]| \leq \sum_{G \in \mathcal{G}} \binom{r}{t}|\mathcal{A}_G| < \sum_{G \in \mathcal{G}} |\mathcal{C}_G| = |\mathcal{C}|,$$

where the strict inequality follows by (6). Hence the result. $\square$

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