Linear Algebraic Equations

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Outline

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Systems of linear equations

- A system of linear equations is a collection of linear equations involving the same set of variables. For example,

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + 0.5y - z &= 0 \end{aligned}$$

- A solution to a linear system is an assignment of numbers to the variables such that all equations are satisfied.
- For the example, a solution is:

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= -2 \end{aligned}$$

General form of system of linear equations

- A general system of m linear equations with n unknowns can be written as:

$$
\begin{aligned}
a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n &= b_1 \\
a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \;\; &\;\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

- $x_1, x_2, \ldots, x_n$ are the unknowns.
- $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the system.
- $b_1, b_2, \ldots, b_m$ are the constant terms.

Matrix form of system of linear equations

- The general system of euqations can be written as a matrix equation of the form:
$$\mathbf{A}\mathbf{x} = \mathbf{b}$$
- $\mathbf{A}$ is an $m \times n$ matrix.
- $\mathbf{x}$ is a column vector with n entries.
- $\mathbf{b}$ is a column vector with m entries.
- $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ can be written as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Solving a system by left division

- A system of linear equations can be written as:

$$\mathbf{Ax} = \mathbf{b}$$

- Pre-multiplying both sides by $\mathbf{A}^{-1}$, we get:

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{Ax} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \end{aligned}$$

- To solve the system, we may type: `>> x = inv(A) * b`
- The inverse can be numerically inaccurate for large systems.
- A better way is to use left division: `>> x = A \ b`
- Left division is performed using Gaussian elimination.

Right division

- Sometimes a system is witten as:
$$\mathbf{xC} = \mathbf{d}$$
where $\mathbf{x}$ and $\mathbf{d}$ are row vectors.
- Post-multiplying both sides by $\mathbf{C}^{-1}$, we get
$$\mathbf{xCC}^{-1} = \mathbf{dC}^{-1}$$
$$\mathbf{x} = \mathbf{dC}^{-1}$$
- To solve the system, we may type: `>> x = d * inv(C)`
- The inverse can be numerically inaccurate for large systems.
- A better way is to use right division: `>> x = d / C`
- This is equivalent to writing: `>> x = C' \ d'`

The determinant

- A system
$$\mathbf{Ax} = \mathbf{b}$$
has m equations and n unknowns.
- $\mathbf{A}$ is an $m \times n$ matrix.
- When $m = n$, $\mathbf{A}$ is a square matrix.
- To check if such a system has a unique solution, we can check the determinant $|\mathbf{A}|$.
- If $|\mathbf{A}| = 0$, the system does not have a unique solution.
- The determinant can be found using: `>> d = det(A)`

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Undetermined systems

- Sometimes, there are more unknowns than equations.
- Consider the system $\mathbf{Ax} = \mathbf{b}$ with m equations and n unknowns.
- $\mathbf{A}$ is an $m \times n$ matrix.
- $\mathbf{x}$ is a column vector with n entries.
- $\mathbf{b}$ is a column vector with m entries.
- When $m < n$, the system is underdetermined.
- Sometimes $m = n$, but some equations are not independent.
- In this case, $|\mathbf{A}| = 0$ and the sytem is underdetermined as well.

The rank of a matrix

- A subdeterminant is the determinant of a matrix generated by eliminating some rows and columns.
- Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -4 & 1 \\ 6 & 10 & 2 \\ 9 & -7 & 3 \end{bmatrix}$$

- The subdeterminant obtained by eliminating the second row and the first column is

$$\begin{vmatrix} -4 & 1 \\ -7 & 3 \end{vmatrix} = -5$$

- The rank of a matrix is the size of the largest subdeterminant that is not zero.
- To find the rank of a matrix, type `>> r = rank(A)`

Existence and uniqueness of solutions

- The augmented matrix is a matrix built from **A** and **b**.
- The augmented matrix is $[\mathbf{A}\ \ \mathbf{b}]$.
- Solutions to a system exist if and only if $\text{rank}[\mathbf{A}] = \text{rank}[\mathbf{A}\ \ \mathbf{b}]$.
- If solutions exist, the solution is unique if $\text{rank}[\mathbf{A}] = n$.
- If solutions exist, but the solution is not unique, there are an infinite number of solutions.
- Let $r = \text{rank}[\mathbf{A}]$.
- r unknown variables can be expressed as linear combinations of the other $n - r$ variables.

Solving underdetermined systems

- Suppose the system $\mathbf{Ax} = \mathbf{b}$ is underdetermined.
- This implies that $r < n$, and that an infinite number of solutions may exist.
- Solving using left division will give a solution with $n - r$ variables set to zero.
- Suppose we have the equation
$$2x_1 + x_2 = 1$$
- We can type:
```
>> A = [2 1];
>> b = 1;
>> x = A \ b;
```
- This gives us $\mathbf{x} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$.

The pseudoinverse

- Sometimes **A** is a square matrix and $|\mathbf{A}| = 0$.
- In this case, the expression `A \ b` will give an error warning us that the matrix **A** is singular.
- In such cases, we can solve using the pseudoinverse method, `pinv`.
- The pseudoinverse method gives the minimum norm solution.
- The Euclidean norm of a vector **x** is $\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.
- We type: `>> x = pinv(A) * b`
- For the equation $2x_1 + x_2 = 1$ we can type:
  ```
  >> A = [2 1]; b = 1;
  >> x = pinv(A) * b;
  ```
- This gives us $\mathbf{x} = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$.

The reduced row echelon form

- Suppose we have a system with rank[$\mathbf{A}$] = rank[$\mathbf{A}\ \mathbf{b}$] = r.
- Suppose that we have n unknowns, and that $r < n$.
- We want to write equations for r unknowns in terms of the other $n - r$ variables.
- To do this, we use the reduced row echelon form function, `rref`.
- First transform $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ such that the required r unknowns are the first elements of $\mathbf{x}$.
- Then use the `rref` function with the augmented matrix:
  `>> Ab2 = rref([A b])`

Example for the reduced row echelon form

- Suppose we have the system of equations:

$$\begin{aligned} x_1 - \quad 2x_2 - \quad x_3 &= -100 \\ 2x_1 + 6.5x_2 + 5x_3 &= \quad 360 \end{aligned}$$

- To express x_1 and x_3 in terms of x_2, rewrite:

$$\begin{aligned} x_1 - \quad x_3 - \quad 2x_2 &= -100 \\ 2x_1 + 5x_3 + 6.5x_2 &= \quad 360 \end{aligned}$$

- This is equivalent to:

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & 5 & 6.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} -100 \\ 360 \end{bmatrix}$$

- ```
  >> A = [1 -1 -2; 2 5 6.5];
  >> b = [-100; 360];
  >> Ab2 = rref([A b]);
  ```
  ```

Example for the reduced row echelon form continued

- Now, `Ab2` is the augmented matrix in reduced row echelon form.

$$\begin{bmatrix} \mathbf{A}_2 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.5 & -20 \\ 0 & 1 & 1.5 & 80 \end{bmatrix}$$

- The system can be written as:

$$\begin{aligned} x_1 + 0x_3 - 0.5x_2 &= -20 \\ 0x_1 + \quad x_3 + 1.5x_2 &= \quad 80 \end{aligned}$$

- This gives us the required equations in terms of x_2:

$$\begin{aligned} x_1 &= 0.5x_2 - 20 \\ x_3 &= -1.5x_2 + 80 \end{aligned}$$

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Overdetermined systems

- An overdetermined system is a system that has more equations than unknowns.
- Some overdetermined systems have exact solutions.
- To check for existence and uniqueness of solutions, use the augmented matrix method described for underdetermined systems.
- When an exact solution exists, `A \ b` returns the exact solution.
- Otherwise, `A \ b` returns a solution that satisfies the system of equations in a least squares sense only.

Example of a system with an exact solution

- Consider the system of equations:

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 + 2x_2 &= 5 \\ 2x_1 + 5x_2 &= 12 \end{aligned} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix}$$

- The rank check can be done as follows:

```
>> A = [1 1; 1 2; 2 5]; b = [3; 5; 12];
>> rA = rank(A);
>> rAb = rank([A b]);
```

- `rA`=2 and `rAb`=2. Since `rA`=`rAb`, an exact solution exists.
- `rA`=2. So a unique solution for x_1 and x_2 exists.
- To find the solution: `>> x = A \ b`
- This gives us:

$$x_1 = 1$$
$$x_2 = 2$$

Example of a system without an exact solution

- Consider the system of equations:

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 + 2x_2 &= 5 \\ 2x_1 + 5x_2 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 10 \end{bmatrix}$$

- The rank check can be done as follows:

```
>> A = [1 1; 1 2; 2 5]; b = [3; 5; 10];
>> rA = rank(A);
>> rAb = rank([A b]);
```

- `rA`=2 and `rAb`=3. Since `rA`≠`rAb`, no exact solution exists.
- To find the solution in the least squares sense: `>> x = A \ b`
- This gives us:

$$\begin{aligned} x_1 &= 1.9091 \\ x_2 &= 1.2727 \end{aligned} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.1818 \\ 4.4545 \\ 10.1818 \end{bmatrix}$$