Real Numbers

1. (a) Give an example of a set of real numbers \( A \) which is bounded above but not bounded below.

(b) Give an example of a set of real numbers \( B \) which has a supremum which is an element of the set \( B \).

(c) Give an example of a set of real numbers \( C \) which has both a supremum and an infimum which are not elements of the set \( C \).

2. Prove that the supremum of a bounded set of real numbers is unique.

3. Show that there is an irrational number between every two real numbers.

4. Write the following real numbers in the form \( \frac{a}{b} \) where \( a, b \) are integers:
   
   0.43\overline{726}, 0.6\overline{131}, 0.9\overline{0215\overline{8}}. \) (The line over the digits means that those digits are recurring.)

5. Let \( A \subseteq B \subseteq \mathbb{R} \). Show that if \( B \) is bounded, then \( A \) is also bounded, and \( \inf(B) \leq \inf(A) \) and \( \sup(A) \leq \sup(B) \). Give examples to show that even if \( A \) is a proper subset of \( B \), equality may hold in the above relations.

6. Let \( A, B \subseteq \mathbb{R} \) be bounded. Let \( C = \{c \in \mathbb{R} \text{ s.t. } c = a + b \text{ for some } a \in A, b \in B\} \). Show that \( \sup(C) = \sup(A) + \sup(B) \). (Hint: use the fact that if \( \alpha = \sup(A) \), then for any \( \epsilon > 0 \), there exists \( a \in A \) s.t. \( \alpha - \epsilon < a \leq \alpha \).)