Introduction to Graph Theory — Sheet 1

Problems marked with an asterisk will be worked out in class.

Elementary

1. * Draw all twenty non-identical graphs on four vertices and three edges. How many of these are non-isomorphic? In general, how many non-identical graphs on \( n \) vertices and \( m \) edges are there? How many non-identical graphs on \( n \) vertices are there?

2. * Let \( \delta = \delta(G) \) and \( \Delta = \Delta(G) \) denote, respectively, the minimum and the maximum degree in the graph \( G \). Show that

\[
\delta \leq \frac{2m}{n} \leq \Delta.
\]

3. * Show that the degrees in a graph cannot all be distinct. (Remember that a graph, unless otherwise stated, has no loops or multiple edges.)

4. An isomorphism \( \phi : V(G) \to V(G) \) is said to be an automorphism of \( G \).

   (a) Show that the set of automorphisms of \( G \) form a group under composition of functions. Denote this group by \( \text{Aut}(G) \).

   (b) Show that \( |\text{Aut}(G)| \) divides \( n! \) and is equal to \( n! \) iff \( G \simeq K_n \) or \( G \simeq \overline{K_n} \).

   (c) Find \( \text{Aut}(G) \) if \( G \) is a 6-cycle.

   (d) Consider the graph \( G \) in Figure 1. How many non-identical labellings of the vertices of \( G \) with the labels \( \{1, 2, 3, 4, 5, 6\} \) are there? What is \( |\text{Aut}(G)|? \) What is the relationship between \( 6! \) and these two results?

![Graph](https://via.placeholder.com/150)

Figure 1: How many distinct labellings does this graph have?
In general, what is the relationship between $|\text{Aut}(G)|$, $n!$ (where $n$ is the number of vertices of $G$), and the number of ways of labelling $G$ with the labels $1, 2, \ldots, n$?

5. * Show that if a graph $G$ is self-complementary, that is, $G \cong \overline{G}$, then $n = 0 \mod 4$ or $n = 1 \mod 4$. Find a self-complementary graph on five vertices.

**Medium**

1. Remember that the distance between two vertices $u, v$ is denoted by $d(u, v)$ and it is equal to the minimum length of a path joining $u$ and $v$. Also, $\delta$ denotes the minimum degree.

   (a) Show that for any $u, v, w \in V$,
   
   $$d(u, w) \leq d(u, v) + d(v, w).$$

   (b) * Show that any two longest paths in a graph must have a common vertex.

   (c) * Show that if $G$ is simple then it must have a path of length $k$ for every $k \leq \delta$.

   (d) * Show that if $G$ is simple and $\delta > \lceil n/2 \rceil - 1$, then $G$ is connected.

   Find a disconnected $\frac{n}{2} - 1$-regular graph for even $n$.

2. * Show that if $G$ is simple and bipartite then

   $$m \leq \frac{n^2}{4}.$$ 

3. Show that in a party of six or more people either there are three persons who know each other or there are three person who are mutual strangers. (Assume that if $x$ knows $y$ then $y$ knows $x$.)

4. * Prove that if $G$ is simple and $\delta \geq 2$ then it contains a cycle of length $\geq \delta + 1$. [Hint: Take a longest path and consider the degree of an endvertex of this path.]

5. Show that if $G$ is simple and connected but not complete then it contains three vertices $u, v, w$ such that $uv, vw \in E(G)$ but $uw \notin E(G)$.

6. Let $c(G)$ denote the number of components of $G$.

   (a) Show that
   
   $$c(G) \leq c(G - e) \leq c(G) + 1$$

   for every edge $e$ in $E(G)$.

   (b) Suggest a similar inequality for $c(G - v)$ where $v$ is a vertex in $V(G)$.  

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(c) * Show that if each degree in $G$ is even and $G$ is disconnected, then there exists no edge $e$ in $E(G)$ such that $G - e$ is disconnected.

(d) Show that if $G$ is connected and each degree is even, then

$$c(G - v) \leq \frac{1}{2} \deg(v),$$

for every vertex $v \in V(G)$.

**Harder**

1. The *girth* $\gamma = \gamma(G)$ of $G$ is the length of a shortest cycle in $G$. If there are no cycles we let $\gamma = \infty$. Prove that

   (a) If $G$ is $r$-regular and $\gamma = 4$ then $n \geq 2r$ and there is exactly one such graph (up to isomorphism) on $2r$ vertices.

   (b) If $G$ is $r$-regular and $\gamma = 5$ then $n \geq r^2 + 1$. Find such a graph for $r = 2, 3$. [Note: It is known that such graphs can only exist if $r = 2, 3, 7$ and possibly $57$.]

2. Let $G$ be simple nd let $p$ be an integer such that $1 < p < n - 1$. Show that if $n \geq 4$ and all induced subgraphs of $G$ on $p$ vertices have the same number of edges, then either $G \simeq K_n$ or $G \simeq \overline{K_n}$.

3. Let $A$ and $B$ be, respectively, the adjacency matrix and the incidence matrix of a graph $G$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A$.

   (a) What is the value of every column sum of $B$? And of $A$

   (b) Show that the number of $[v_i, v_j]$-walks of length $k$ in $G$ is given by the $i, j$-entry of $A^k$.

   (c) Show that if $G$ is simple, then the entries on the diagonals of both $BB^t$ and $A^2$ are the degrees of the vertices of $G$.

   (d) Why is each eigenvalue of $A$ real?

   (e) Show that

   i. $\sum \lambda_i = 0$.

   ii. $\sum \lambda_i^2 = 2m$, where $m$ is the number of edges of $G$.

   iii. $\sum \lambda_i^3 = 6t$, where $t$ is the number of triangles of $G$.

   iv. For each $\lambda_i$, $|\lambda_i| \leq \sqrt{2m(n - 1)}/n$. 

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