1. (a) Let $A_1, A_2, \ldots, A_n$ be finite sets, and let $\alpha_i$ denote the sum of the cardinalities of the intersections of the sets taken $i$ at a time. Prove that

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} \alpha_i (-1)^{i+1}.$$ 

(b) How many integers are there in the range 1 to 999 (inclusive) which are not divisible by any of 2, 5 or 23?

2. (a) Write down the coefficient of $x^k$ in the expansion of $(1 - x)^{-n}$ in ascending powers of $x$.

(b) Find the coefficient of $x^{20}$ in each of the following products:

(i) $(x^2 + x^3 + x^4 + \ldots)^4$

(ii) $(x^2 + x^3 + \ldots + x^{12})^4$

(iii) $(x^2 + x^3 + x^4 + \ldots)^3(x + x^2 + \ldots + x^6)$

3. (a) A loan of Lm3000 is taken from a bank. After a year, and at the end of every subsequent year, a repayment of Lm$P$ is effected. Moreover, at the end of every year the bank charges interest at the rate of 1 per cent of the amount owed during that year.

   Let $A_n$ denote the amount owed to the bank at the end of the $n$th year (therefore $A_0 = 3000$). Obtain and solve a recurrence relation for $A_n$.

(b) Solve the recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = n5^n \quad (n \geq 0)$$
given that \( a_0 = a_1 = 0 \).

4. (a) Let \( p(n) \) denote the number of partitions of the positive integer \( n \) and let \( p(n|\mathcal{P}) \) denote the number of partitions satisfying a given property \( \mathcal{P} \).

   (i) Write down the generating function for \( p(n) \).

   (ii) Let \( k \geq 1 \) be fixed, let \( \mathcal{P}_1 \) be the property “No part in the partition appears more than \( k \) times” and let \( \mathcal{P}_2 \) be the property “No part in the partition is divisible by \( k + 1 \)”. Prove that \( p(n|\mathcal{P}_1) = p(n|\mathcal{P}_2) \).

(b) Let \( S(n,k) \) denote the number of ways of partitioning an \( n \)-set into \( k \) parts. Write down a recurrence relation for \( S(n,k) \) and use this relation to find all values of \( S(n,k) \) for \( 1 \leq k \leq n \leq 5 \).

   How many ways are there to distribute 5 distinguishable balls amongst 6 distinguishable boxes such that exactly three of the boxes are nonempty? [Hint: For each choice of the three nonempty boxes, each distribution is a surjection from a 5-set to a 3-set.]