Section B

4. Let $G$ be a graph on $n$ vertices and let $A$ be the adjacency matrix of $G$. Let $\pi$ be a permutation of $V(G)$ and $P$ the permutation matrix representing $\pi$. Prove that $\pi$ is an automorphism of $G$ if and only if $PA = AP$. [7 marks]

Suppose, from now on, that $\pi$ is an automorphism of $G$. Show that if $x$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$, then $Px$ is also an eigenvector of $A$ corresponding to $\lambda$. [3 marks]

Deduce that if $\lambda$ is a simple eigenvalue of $A$ and $x$ is a corresponding eigenvector with real components, then $Px = \pm x$. [7 marks]

Hence show that if all eigenvalues of $G$ are simple, then every non-trivial automorphism of $G$ has order 2. [7 marks]

Suppose now that the permutation $\pi$ has $s$ cycles of odd length and $t$ cycles of even length, when written as a product of disjoint cycles. Prove that the number of simple eigenvalues of $G$ is at most $s + 2t$. [10 marks]

5. (a) Show that if the automorphism group of a graph $G$ contains a sub-permutation group $\Gamma$ acting regularly on $V(G)$ then $G$ is a Cayley graph of $\Gamma$ with respect to some $S \subseteq \Gamma$ with $S = S^{-1}$. [10 marks]

(b) Let $\Gamma$ be a finite group and $S \subseteq \Gamma$ a generating set of $\Gamma$ with $S = S^{-1}$. Let $G$ be the Cayley graph of $\Gamma$ with respect to $S$. Show that if $\phi$ is an automorphism of $\Gamma$ such that $\phi(S) = S$, then $\phi$ is also an automorphism of the graph $G$. [7 marks]

(c) Now let $G$ be any graph, and let $\Gamma = \text{Aut}(G)$. Suppose $\Gamma$ is abelian and acts transitively on $V(G)$. Show that $\Gamma$ is an elementary abelian 2-group. [You may assume that if an abelian subgroup of $S_Y$ (the group of all permutations on $Y$) acts transitively on $Y$ then it also acts regularly.] [7 marks]

(d) Let $G$ be a connected cubic Cayley graph $\text{Cay}(\Gamma, S)$ of an abelian group $\Gamma$, and suppose that $G$ has order $2n$ which is greater than 8. Show that the elements of $S$ are of the form $\alpha, \beta, \beta^{-1}$ where the order of $\alpha$ is 2 and that of $\beta$ is greater than two. Show also that the order of $\beta$ is either $n$ or $2n$. [10 marks]

6. (a) Let $G$ be a graph without isolated vertices. Show that the deck of $G$ is uniquely determined from the edge-deck of $G$. [Any form of Kelly’s Lemma, if required, may be quoted without proof.] [12 marks]

(b) Assuming that the minimum degree $\delta$ of a graph $G$ is reconstructible from its edge-deck, show that a graph $G$ is edge-reconstructible in each of the following cases.
(i) $G$ contains two adjacent vertices both of degree $\delta$; [2 marks]

(ii) $G$ contains a $(\delta + 1)$-vertex adjacent to two $\delta$-vertices; [5 marks]

(iii) $G$ contains a triangle with one $\delta$-vertex and two $(\delta + 1)$-vertices. [3 marks]

(c) A graph is said to have property $EA_k$ if $G - A \neq G - B$ for any two distinct subsets $A, B$ of $E(G)$ with $|A| = |B| = k$. Prove that if $G$ has property $EA_3$ then it can be reconstructed from any two edge-deleted subgraphs in its edge-deck. [12 marks]