Astrometrics

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1 Time

The time $t$ measured by an inertial observer in the solar system’s reference is called **universal time** (or GMT\(^1\)) (assuming no relativistic effects). Two time intervals are important:

1. The average time it takes for the Earth to orbit once around the Sun (or as seen from Earth, for the Sun to go round the *ecliptic*) is called a **year**, so the average angular speed of the Earth around the Sun is

$$\omega_{\odot} = \frac{360^\circ}{\text{year}}$$

This is more properly called the **sidereal** year because the angular speed is measured against the backdrop of ‘fixed’ stars.

2. The average time it takes for the Sun to appear in the same place after one rotation of the Earth is the

**day** = 24 **hours** = 24 × 60 **minutes** = 24 × 60 × 60 **seconds**.

It is a bit longer than the time it takes the Earth to rotate 360°, because it has to make up for the fact that the Earth has gone round the Sun an extra angle.

In general, the resultant of two rotations is $\omega_{\text{res}} = \omega_1 + \omega_2$, or equivalently

$$\frac{1}{T_{\text{res}}} = \frac{1}{T_1} + \frac{1}{T_2}.$$ 

The average angular speed of the Earth’s rotation is

\(^1\)Strictly speaking, physical time as measured by atomic clocks is called International Atomic Time. Universal Time counts the same seconds, but inserts a leap second every few years to keep in pace with the slowing day; so the difference between Atomic Time and Universal Time is of a few seconds over decades.
One does not expect these two time intervals to be related in any nice way,

\[ 1 \text{ sidereal year} = 365.25636 \text{ days} \]

So, adding \( 360^\circ/\text{day} = 15^\circ/\text{hour} = 15'/\text{min} = 15''/\text{sec} \),

\[ \omega_\odot = \frac{360^\circ}{\text{year}} = 0.986^\circ/\text{day} = 2.464'/\text{hour} = 2.464''/\text{min} \]
\[ \omega_E = 15^\circ.2.28''/\text{hour} = 15.041''/\text{sec} \]

Three complications arise because the Earth’s axis precesses (rotates) once every 25771 years, the orbital perihelion rotates every 113,000 years, and its daily rotational period is slowly and irregularly increasing. From an Earth perspective, the time it takes from one midsummer to the next, called the tropical year, is slightly less than a sidereal year as

\[ \frac{1}{T_{\text{trop.year}}} = \frac{1}{T_{\text{sid.year}}} + \frac{1}{T_{\text{prec.}}} = \frac{1}{365.2419\text{days}} \]

### 1.1 Calendar

The Gregorian calendar is set up to approximate the tropical year as closely as possible over millennia: \( 365.24219 \approx 365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400} \)

A leap day is added to every year divisible by 4; unless the leap year is the start of a century; unless that century is divisible by 4.

This gives a civil year of 365.25 days in the short term, and 365.2425 days in the long term.

#### 1.1.1 Changing from Date to Time

The number of days from start 1 January until the start of date D/M is

\[
\text{days} = \begin{cases} 
D - 1 & \text{if January} \\
D + 30 & \text{if February} \\
D + \lfloor 30.6(M + 1) \rfloor - 64 + \chi_{\text{leap}} & \text{if } \geqslant \text{March}
\end{cases}
\]

where \( \chi_{\text{leap}} \) is 1 if the year is a leap year, and 0 otherwise

\[ \chi_{\text{leap}} = 1 - \lfloor y/4 \rfloor + \lfloor y/4 \rfloor + \lfloor y/100 \rfloor - \lfloor y/100 \rfloor - \lfloor y/400 \rfloor + \lfloor y/400 \rfloor \]
The number of days from 2000.0 until the year $2000 + Y$ (including it) is, taking $y = Y - 1$,

$$366 + 365y + \lfloor y/4 \rfloor - \lfloor y/100 \rfloor + \lfloor y/400 \rfloor$$

The **Local Time** is related to universal time by

$$\text{Local Time} = t + \text{“time zone”} + \text{“daylight saving”}$$

For example, the total time in days since 2000.0 of the date 2024 Sep 5 1022:43 (GMT) is given by: number of days 2000–2023 is 8766, Jan–Aug is 244, plus 4 and $(10 + (22 + 43/60)/60)/24$, in total 9009.43244 days.

### 1.2 Sidereal ‘Time’

**Sidereal Time** is the angle that Earth’s $0^\circ$ longitude (Greenwich) is pointing at on the celestial equator. It is not really a time, but it is traditional to measure celestial angles using hours, minutes, and seconds; a great circle on the celestial sphere is divided into 24 angular ‘hours’, so one angular hour equals $15^\circ$.

$$\text{Sidereal Time GST} = \theta_0 + \omega_E(t - t_0)$$

where $\theta_0$ is the direction of Greenwich at $t_0$; e.g. on midnight 1 Jan 2000, $\theta_0 = 99.967795^\circ$, on midnight 1 Jan 2010 $\theta_0 = 100.537624^\circ$.

Local Sidereal Time is the direction that a particular longitude $\phi$ on Earth is pointing at, i.e., if you imagine a line going from North to South passing directly overhead, LST is the angle between the celestial $0^\circ$ and this imaginary line,

$$\text{LST} = \text{GST} + \text{longitude}$$

An angle $\alpha$ on the celestial equator is at LST $- \alpha$ away from this imaginary line.
To convert from LST to local time,

\[
\text{Local Time} = \frac{\text{LST} - \text{long.} - \theta_0}{\omega_E} + t_0 + \text{“time zone”} + \text{“daylight saving”}
\]

2 Spherical Coordinates

The standard spherical coordinates are akin to the longitude and latitude that are used for positions on the Earth’s surface; such coordinates are fully characterized by a great circle (the equator) and a reference point on it (the 0 longitude).

\[
r = \begin{pmatrix} 
\cos \theta \cos \phi \\
\cos \theta \sin \phi \\
\sin \theta 
\end{pmatrix}
\]

There are various spherical coordinates projected to the celestial sphere:

<table>
<thead>
<tr>
<th>Name</th>
<th>Coordinate names</th>
<th>Reference circle and point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equatorial</td>
<td>((\alpha, \delta))</td>
<td>celestial equator, ‘first point of Aries’</td>
</tr>
<tr>
<td>Altazimuth</td>
<td>(A, a)</td>
<td>horizon, north</td>
</tr>
<tr>
<td>Ecliptic</td>
<td>((\lambda, \beta))</td>
<td>ecliptic, ‘first point of Aries’</td>
</tr>
<tr>
<td>Galactic</td>
<td>(l, b)</td>
<td>Milky Way, its center</td>
</tr>
</tbody>
</table>

The ‘first point of Aries’ is, by definition, the point where the equator intersects the ecliptic in the constellation of Aries. Altazimuth coordinates of points on the celestial sphere are local and continually changing. The rest are relatively fixed but have slight problems: Earth’s rotational axis and equator move slowly (precession, 25ky); the ecliptic plane also precesses very slowly (430My) due to perturbations from the other planets; the galactic plane is hard to define precisely.
2.1 Changing Coordinates

2.0.1 Angle between two Directions

The *elongation* \( \gamma \), or angular difference, between two coordinates \((\alpha_1, \delta_1)\) and \((\alpha_2, \delta_2)\) is given by

\[
\cos \gamma = \left( \cos \delta_1 \cos \alpha_1 \sin \delta_1 \right) \cdot \left( \cos \delta_2 \cos \alpha_2 \sin \delta_2 \right) = \cos(\alpha_1 - \alpha_2) \cos \delta_1 \cos \delta_2 + \sin \delta_1 \sin \delta_2
\]

2.1 Changing Coordinates

To change from one spherical coordinate system \((\phi, \theta)\) to another \((\alpha, \delta)\), consider their reference great circles and let \(i\) be a vector where they intersect. Suppose the reference points of the two systems have angles \(\phi_0\) and \(\alpha_0\) from \(i\). Let \(i, j, k\) be a right-handed orthonormal vector basis with \(j\) in the plane of the \((\phi, \theta)\) reference circle, and \(i, b, c\) a right-handed orthonormal basis with \(b\) in the plane of the \((\alpha, \delta)\) reference circle. If the two planes are at an inclination of \(\epsilon\) to each other, then the relation between the two bases is

\[
b = \cos \epsilon \ j - \sin \epsilon \ k \\
c = \sin \epsilon \ j + \cos \epsilon \ k
\]

So the relation between the coordinates of a point on the sphere, with respect to the two bases \((i, b, c)\), \((i, j, k)\), is

\[
\begin{pmatrix}
\cos \delta \cos(\alpha - \alpha_0) \\
\cos \delta \sin(\alpha - \alpha_0) \\
\sin \delta
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \epsilon - \sin \epsilon & 0 \\
0 & \sin \epsilon & \cos \epsilon
\end{pmatrix}
\begin{pmatrix}
\cos \theta \cos(\phi - \phi_0) \\
\cos \theta \sin(\phi - \phi_0) \\
\sin \theta
\end{pmatrix}
\]

Dotting with the various unit vectors, one obtains the following identities:

\[
\begin{align*}
(i) & \quad \cos(\alpha - \alpha_0) \cos \delta = \cos(\phi - \phi_0) \cos \theta \\
(b) & \quad \sin(\alpha - \alpha_0) \cos \delta = \sin(\phi - \phi_0) \cos \theta \cos \epsilon - \sin \theta \sin \epsilon \\
(c) & \quad \sin \delta = \sin(\phi - \phi_0) \cos \theta \sin \epsilon + \sin \theta \cos \epsilon \\
(j) & \quad \sin(\phi - \phi_0) \cos \theta = \sin(\alpha - \alpha_0) \cos \delta \cos \epsilon + \sin \delta \sin \epsilon \\
(k) & \quad \sin \theta = -\sin(\alpha - \alpha_0) \cos \delta \sin \epsilon + \sin \delta \cos \epsilon
\end{align*}
\]

Dividing the first two equations, one gets

\[
\frac{\sin \delta = \sin(\phi - \phi_0) \cos \theta \sin \epsilon + \sin \theta \cos \epsilon}{\sin(\alpha - \alpha_0) \cos \delta \sin \epsilon + \sin \delta \cos \epsilon}
\]  \hspace{1cm} (1)

\[
tan(\alpha - \alpha_0) = \frac{\tan(\phi - \phi_0) \cos \epsilon - \sec(\phi - \phi_0) \tan \theta \sin \epsilon}{\tan \theta \sin \epsilon} \hspace{1cm} (2)
\]
2.2 Rising and Setting Times for \((\alpha, \delta)\)

Other useful identities are

\[
\sin(\alpha - \alpha_0) = \frac{\sin \delta \cos \epsilon - \sin \theta}{\cos \delta \sin \epsilon} \\
\tan(\alpha - \alpha_0) = \frac{\sin \delta \cos \epsilon - \sin \theta}{\cos \theta \cos(\phi - \phi_0) \sin \epsilon}
\]

To convert from \((\alpha, \delta)\) to \((\phi, \theta)\), use the same identities with \(\epsilon\) replaced by \(-\epsilon\).

Some useful reference data:

<table>
<thead>
<tr>
<th>Altazimuth → Equatorial</th>
<th>(\epsilon = \phi - 90^\circ)</th>
<th>(\alpha_0 = \text{LST} - 90^\circ)</th>
<th>(A_0 = 90^\circ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ecliptic → Equatorial</td>
<td>(\epsilon = 23^{\circ}26'16'')</td>
<td>(\alpha_0 = 0^\circ)</td>
<td>(\lambda_0 = 0^\circ)</td>
</tr>
<tr>
<td>Galactic → Equatorial</td>
<td>(\epsilon = 62^{\circ}52'18'')</td>
<td>(\alpha_0 = 282^{\circ}51'34'')</td>
<td>(l_0 = 32^{\circ}55'55'')</td>
</tr>
</tbody>
</table>

where \(\phi\) is the place’s latitude.

Earth’s obliquity (angle between equator and ecliptic) changes (due to the other planets) as

\[
\epsilon \approx 23.439291^\circ - 0.8325^\circ \sin(T/6523) - 0.00675^\circ \sin(T/2778) \pm 0.0002^\circ
\]

where \(T\) is the number of years since 2000.

2.2 Rising and Setting Times for \((\alpha, \delta)\)

These are the times when \((\alpha, \delta)\) have an altitude of 0. Substituting into (1) for a latitude \(\phi\),

\[
0 = \sin a = \sin(\alpha - \text{LST} + 90^\circ) \cos \delta \sin(90^\circ - \phi) + \sin \delta \cos(90^\circ - \phi) \\
= \cos(\alpha - \text{LST}) \cos \delta \cos \phi + \sin \delta \sin \phi \\
\therefore \text{LST} = \alpha \pm \cos^{-1}(-\tan \delta \tan \phi)
\]

The local sidereal time can then be converted to local time.

The corresponding azimuth at rising and setting is given by

\[
\sin(A - 90^\circ) = \frac{\sin 0 \cos(90^\circ - \phi) - \sin \delta}{\cos 0 \sin(90^\circ - \phi)} \\
\Rightarrow \cos A = \frac{\sin \delta}{\cos \phi}
\]

In practice, one needs to correct for atmospheric refraction, which increases the altitude.

Sunrise/set occurs when the Sun has an altitude of about \(a = -0.83^\circ\) (accounting for refraction), twilight until it is about \(-6^\circ\), getting dark \((-6^\circ\) to \(-12^\circ\)), until completely dark at night \((-18^\circ\)). Similarly, moonrise/set occurs at
an altitude of $-0.83^\circ$. For these cases, use the more general formula that gives the time when $(\alpha, \delta)$ is at an altitude $a$:

$$LST = \alpha \pm \cos^{-1}(\sin a \sec \delta \sec \phi - \tan \delta \tan \phi)$$

If the point is moving (e.g. Moon) then calculate $(\alpha, \delta)$ at two times (separated by $\delta t$), and find their corresponding rising/setting times $T_1, T_2$. The correct rising/setting time is then the point of intersection $t = T_1 + (T_2 - T_1) t/\delta t$. If the object is near (e.g. Moon), use the observer-corrected coordinates $(\alpha, \delta)$ (see below, parallax).

### 2.2.1 Refraction

The altitude angle $a$ is refracted to $a'$ by Snell’s law

$$\sin z' = \frac{\sin z}{1.000282}$$

where $z = 90^\circ - a$ is the zenith angle (the refractive index depends on the light frequency).

### 2.2.2 Parallax

For nearby objects such as the Moon, the observed equatorial coordinates differ from the geocentric coordinates $(\alpha, \delta)$. If the distance of the object from the Earth’s center is $r$, Earth’s radius is $a$, and the observer has latitude $\phi$, then the observed coordinates $(\alpha', \delta')$ and distance $\rho$ are

$$\rho \left( \frac{\cos \delta' \cos \alpha'}{\sin \delta'} \right) = r \left( \frac{\cos \delta \cos \alpha}{\sin \delta} \right) - a \left( \frac{\cos \phi \cos LST}{\sin \phi} \right)$$

### 2.3 Precession

Like any symmetric top, the Earth precesses at a rate $\omega_{pre}$ where

$$\frac{3G}{2} \left( \frac{M_\odot}{r_\odot^3} + \frac{M_m}{r_m^3} \right) (C - A) \cos \epsilon \sin \epsilon = \text{torque} = \omega_{pre} \omega_E C \sin \epsilon$$

For the oblate Earth, $A/C = 0.9967$, so $\omega_{pre} \approx 0.0142^\circ$/year. In fact, the equatorial plane is rotating about the ecliptic north at a rate of $360^\circ/25771$ years = $50.23^\prime\prime$/year, e.g. after more than two millennia, the first point of Aries is now in Pisces. So the ecliptic coordinates of a star are changing at the rate of $\dot{\lambda} = 50.23^\prime\prime$/year, $\dot{\beta} = 0$. Differentiating (1) and (2) gives

$$\dot{\delta} \cos \delta = \cos \beta \sin \epsilon \dot{\lambda}$$

$$-\sin \alpha \cos \delta \dot{\alpha} - \cos \alpha \sin \delta \dot{\delta} = -\sin \lambda \cos \beta \dot{\lambda}$$
\[ \dot{\delta} = \cos \alpha \sin \epsilon \dot{\lambda} = 19.98'' \cos \alpha/\text{year}, \]
\[ \dot{\alpha} = (\cos \epsilon + \sin \alpha \tan \delta \sin \epsilon) \dot{\lambda} \]
\[ = (46.09'' + 19.98'' \sin \alpha \tan \delta)/\text{year} \]

Note that there is a separate tiny precession (wobble) of a few metres every 1.18 years, due to the Earth’s rotation not being exactly through the North-South axis (i.e., the moment of inertia symmetry axis).

3 Orbits

Consider the path of an object orbiting the Sun. Relative to the center of mass of the system, \( m \mathbf{r} + M_\odot \mathbf{R}_\odot = 0 \), hence \( \mathbf{r} - \mathbf{R}_\odot = (1 + \frac{m}{M_\odot}) \mathbf{r} \). The force acting on the object is
\[
m \ddot{\mathbf{r}} = -\frac{GM_\odot m}{|\mathbf{r} - \mathbf{R}_\odot|^2} \dot{\mathbf{r}} = -\kappa m \frac{\dot{\mathbf{r}}}{|\mathbf{r}|^2}
\]
where \( \kappa = GM_\odot (1 + m/M_\odot)^{-2} \). By conservation of angular momentum (the force is independent of angle), the motion is in a plane, so polar coordinates \((r, \theta)\) suffice, and the equations reduce to \( r^2 \dot{\theta} = h \) (constant) and \( \ddot{r} - r \dot{\theta}^2 = -\frac{\kappa}{r^2} \).

Combining the two and substituting \( u := 1/r \) gives \( u'' + u = \frac{\kappa}{r^2} = \alpha \) whose solution
\[
r = \frac{\alpha}{1 + e \cos(\theta - \theta_0)}
\]
is a conic.

It is an ellipse when the eccentricity \( e < 1 \), with \( \alpha = a(1 - e^2) \), \( a \) the semi-major axis, and \( \theta_0 \) the direction of the point closest to the focus, called the perihelion. Substituting into \( r^2 \dot{\theta} = h \), a differential equation is obtained for the “true anomaly” \( \nu = \theta - \theta_0 \) measured from the perihelion
\[
\dot{\nu} = \beta (1 + e \cos \nu)^2
\]
where \( \beta = \frac{r^2}{2} = \frac{2\pi}{T^2} = 2\pi (1 - e^2)^{-3/2} \), \( T = \int_0^{2\pi} \frac{d\nu}{\sqrt{\kappa a^{3/2}}} = \frac{2\pi a^{3/2}}{\sqrt{\kappa}} \) is the period of the orbit (Kepler’s third law). This is a separable type equation,
\[
\int \frac{d\nu}{(1 + e \cos \nu)^2} = \beta t
\]
To solve, it is best to change variables from $\nu$ to the “eccentric anomaly” $E$ defined by

$$\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2} \quad (4)$$

After some calculations, the equation becomes

$$\dot{E} = \frac{2\pi}{T} \frac{1}{1-e \cos E}$$

Changing coordinates again from $E$ to the “mean anomaly” (Kepler)

$$M := E - e \sin E \quad (5)$$
gives the simplified equation $\dot{M} = 2\pi/T$, i.e.,

$$M(t) = \frac{2\pi}{T}(t - t_0) + M(t_0) \quad (6)$$

Sometimes, the reference time $t_0$ is that of perihelion, in which case $M(t_0)=0$. More often, angles are measured from a reference direction; either this is the node where the orbital plane cuts the ecliptic, $M(t_0) = \varepsilon - \omega$, where $\varepsilon$ is the mean anomaly and $\omega$ is the perihelion angle; or the reference direction is the first point of Aries and $M(t_0) = L - \omega'$ where $L = \varepsilon + \Omega$, $\omega' = \omega + \Omega$.

Orbital characteristics: given an object of mass $m$ orbiting another $M$ in an ellipse of eccentricity $e$ and semi-major axis $a$, the period $T$, angular momentum $m\ell$, total energy $E$, maximum and minimum speeds $v_\pm$, and peri/aphelion distances $a_\pm$, are given by

$$T^2 = \frac{4\pi^2}{\kappa} a^3, \quad \ell^2 = \kappa a(1-e^2), \quad E = -\frac{\kappa m}{2a},$$

$$v_\pm = \frac{\kappa (1 \pm e)}{a (1 \mp e)}, \quad a_\pm = (1 + \frac{m}{M})(1 \pm e)a$$

Note that these formulæ apply only to a two-body orbit; the Sun has several orbiting planets, so to a good approximation it can be taken to be at the origin, and the factor $(1 + m/M)$ can be ignored $(\kappa = GM_\odot)$.

3.1 Calculating the position

To find the position $(\alpha, \delta)$ of an orbiting object (planet), as seen from Earth:

1. Find the mean anomaly $M(t)$, using (6) and $t, t_0, T$. 
2. Find the eccentric anomaly \( E(t) \), using (5) and \( M(t) \), \( e \); since the equation has no closed-formula solution, a Newton-Raphson or other iterative method is used.

3. Find the true anomaly \( \nu(t) \), using (4) and \( E(t) \), \( e \).

4. Find the distance \( r(t) \), using (3) and \( \nu(t) \), \( e \), \( a \).

5. The heliocentric coordinates with respect to the orbit’s plane are: longitude \( l' = \nu + \omega = \nu + \omega' - \Omega \) measured from the node, and latitude = 0; if the planet is Earth and the Sun’s position is sought, then go straight to step 8, using \( \lambda = l' + 180^\circ \), \( \beta = 0 \) (This is not extremely accurate, because it is the Moon-Earth system that moves in an elliptical orbit.)

6. The heliocentric coordinates with respect to the ecliptic plane can then be found by changing coordinates from \((l', 0)\) to \((l, \tau)\); the angle between the object’s orbit and the ecliptic is the inclination \( \epsilon := i \) of the orbit, while the longitude reference points are \( l'_0 = 0 \), \( l_0 = \Omega \);

\[
\begin{align*}
\sin \tau &= \sin l' \sin i \\
\tan(l - \Omega) &= \tan l' \cos i
\end{align*}
\]

7. The geocentric ecliptic coordinates \((\lambda, \beta)\) are obtained from \((l, \tau)\) using the next section, after having first calculated \( l_E \) and \( r_E \) for Earth.

8. The equatorial coordinates \((\alpha, \theta)\) are found from \((\lambda, \beta)\).

3.1.1 Changing the Center of a Coordinate System

That is, from a heliocentric ecliptic system \((l, \tau)\) to the geocentric ecliptic \((\lambda, \beta)\).
\[ r \sin \tau = \rho \sin \beta \]
\[ r_E = r \cos \tau \cos(l - l_E) + \rho \cos \beta \cos(180^\circ + l_E - \lambda) \]
\[ r \cos \tau \sin(l - l_E) = \rho \cos \beta \sin(180^\circ + l_E - \lambda) \]

So
\[ \tan(L - \lambda) = \frac{r \cos \tau \sin(l - l_E)}{r_E - r \cos \tau \cos(l - l_E)} \]
\[ \tan \beta = \frac{\tan \tau \sin(\lambda - l_E)}{\sin(l - l_E)} \]
\[ \rho^2 = r_E^2 + r^2 - 2r_E r \cos \tau \cos(l - l_E) \]

give the values of \( \lambda, \beta, \rho \) from \( l, \tau, r \).

The phase of the object is the fraction of the amount of light reflected from the object. By Lambert’s law, the amount of diffuse reflection from a surface with normal \( \mathbf{n} \) in the direction \( \mathbf{v} \) is \( \mathbf{v} \cdot \mathbf{n} \); for a spherical object with incident light from \((\cos \alpha, \sin \alpha, 0)\), the total reflected in the direction \( \mathbf{i} \) is

\[
\int_{\alpha - \pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left( \begin{array}{c} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ \sin \theta \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \cos \theta \, d\theta \, d\phi = \frac{\pi}{2} (1 + \cos \alpha)
\]

so the phase is
\[ F = \frac{(1 + \cos \alpha) \pi/2}{(1 + 1) \pi/2} = \frac{1 + \cos \alpha}{2} \]

For an astronomical object, \( \alpha \) is the angle Sun-Object-Earth, i.e., between the spherical coordinates \((l, \tau)\) and \((\lambda, \beta)\):
\[ \cos \alpha = \cos \tau \cos \beta \cos(l - \lambda) + \sin \tau \sin \beta \]

Its angular size is
\[ \text{angular size (in radians)} = \frac{\text{diameter of object}}{\rho} \]

Orbital data of planets at 2000 (with change every 10 years)
3.1 Calculating the position

\[ a \text{ (AU)} \quad e \quad i \text{ (°)} \quad L \text{ (°)} \quad \omega_p \text{ (°)} \quad t_0 \quad \Omega \text{ (°)} \]

<table>
<thead>
<tr>
<th>Planet</th>
<th>( a ) (AU)</th>
<th>( e )</th>
<th>( i ) (°)</th>
<th>( L ) (°)</th>
<th>( \omega_p ) (°)</th>
<th>( t_0 ) (2000)</th>
<th>( \Omega ) (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>0.387099927</td>
<td>0.20563593</td>
<td>7.60497902</td>
<td>252.25632350</td>
<td>77.4579628</td>
<td>2000.1280</td>
<td>48.33076593</td>
</tr>
<tr>
<td>Venus</td>
<td>0.72333566</td>
<td>0.00677672</td>
<td>3.39467605</td>
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<td>Earth-Moon</td>
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<td>0.05386179</td>
<td>2.48599187</td>
<td>49.95424423</td>
<td>92.59887381</td>
<td>2003.5304</td>
<td>113.66242448</td>
</tr>
<tr>
<td>Uranus</td>
<td>19.18916464</td>
<td>0.04725744</td>
<td>0.77263783</td>
<td>313.23810451</td>
<td>170.95427630</td>
<td>2050.7349</td>
<td>74.01692503</td>
</tr>
<tr>
<td>Neptune</td>
<td>30.06992276</td>
<td>0.00859048</td>
<td>1.77004372</td>
<td>-55.12002969</td>
<td>44.96746277</td>
<td>2045.6635</td>
<td>131.78422574</td>
</tr>
</tbody>
</table>

1 AU = 149,597,871 km; the period of a planet is equal to \( a^{3/2} \) if \( a \) is measured in AU. Earth’s perihelion rotates 360° in 11,110,000 years. Sun’s diameter is \( 1.392 \times 10^6 \) km.

Data from ssd.jpl.nasa.gov. For asteroids etc. see www.minorplanetcenter.org/iau/MPCORB.html, ssd.jpl.nasa.gov/sbdb.cgi, or asteroid.lowell.edu. It is hard to track satellites’ orbits because they are easily perturbed, but see www.amsat.org/amsat-new/tools.

Check calculations with www.jb.man.ac.uk/almanac

3.1.2 Parabolic and Hyperbolic Orbits

For hyperbolic orbits \( e > 1 \) (e.g. near Earth asteroids/meteoroids), the equation \( \dot{\theta} = \beta (1 + e \cos \theta)^2 \) can be solved by applying the change of variable,

\[
\tanh \frac{E}{2} = \sqrt{\frac{e - 1}{e + 1}} \tan \frac{\theta}{2}
\]

to get

\[
\dot{E} = \frac{\beta (e^2 - 1)^{3/2}}{e \cosh E - 1}
\]

A second change of variables, \( M = e \sinh E - E \) leads to \( \dot{M} = \beta (e^2 - 1)^{3/2} t \), so

\[
M(t) = \beta (e^2 - 1)^{3/2} t
\]

where \( \beta^2 = \frac{\kappa}{q^{(1+e)}\kappa} \), since the perigee distance is \( q = h^2/\kappa(1+e) \) and \( \beta = \kappa^2/h^3 \).

For parabolic orbits \( e = 1 \) (e.g. comets), the equation \( \dot{\theta} = \beta (1 + \cos \theta)^2 \) can be solved directly,

\[
\beta t = M = \int \frac{d\theta}{(1 + \cos \theta)^2} = \frac{\sin \theta (2 + \cos \theta)^3}{3(1 + \cos \theta)^2} = \frac{1}{6} \tan^3 \frac{\theta}{2} + \frac{1}{2} \tan \frac{\theta}{2}
\]

hence \( x := \tan \theta/2 \) satisfies a cubic equation, so

\[
\tan \frac{\theta}{2} = (3M + \sqrt{9M^2 + 1})^{1/3} - (3M + \sqrt{9M^2 + 1})^{-1/3}
\]
3.2 Finding an Orbit from Observations

An orbit requires 6 parameters to be determined in space: \( a, e, i, \omega, \Omega, M(t_0) \), so three observations \((\alpha(t_i), \delta(t_i))\) should in principle be enough.

There are errors in the calculation: light takes time to arrive to Earth, and the observations are from a point on the surface of the Earth, not its center.

3.3 Lunar Orbit

The Moon’s motion is much more complicated because of the gravitational effect of the Earth and the Sun:

\[
\ddot{r}_E = -\frac{GM_\odot r_E}{|r_E|^3} - \frac{GM_m r}{|r|^3}
\]
\[
\ddot{r}_m = -\frac{GM_\odot r_m}{|r_m|^3} + \frac{GM_\odot r}{|r|^3}
\]

where \( r = r_m - r_E \). The Earth’s motion is dominated by the Sun’s attraction, so can be taken to be an elliptic orbit, even a circular orbit to a good approximation \( r_E = a(\cos \theta_E, \sin \theta_E, 0) \); but the Moon’s is dominated by both the Sun (69\%) and Earth (31\%). In general, the three-body problem has no simple solution. If only the leading terms in \( r/|r_E| \) are kept, letting \( G(M_E + M_m) = \omega_m^2 a_m^3 \), \( GM_\odot = \omega_E^2 a^3 \), \( m := \omega_\odot/\omega_m \), and changing scales, \( \mathbf{R} := r/a_m, T := \omega_m t \), simplifies the second equation to

\[
\ddot{\mathbf{R}} = -\frac{\mathbf{R}}{|\mathbf{R}|^3} + m^2 (3\mathbf{R} \cdot \dot{\mathbf{R}} \dot{\mathbf{R}}_E - \mathbf{R})
\]

The dominant part is \( \ddot{\mathbf{R}} = -\mathbf{R}/|\mathbf{R}|^3 \), i.e., an elliptical orbit around the Earth in a plane inclined at an angle \( i \) to the ecliptic, eccentricity \( e \), and with mean angular speed \( \omega_m = 360^\circ/27.321662 \text{ days} (=27.321582 \text{ days with respect to the precessing ecliptic coordinates}) \). But this behavior is modified by the remaining term; the real orbit is not planar: both the ellipse and the plane are slowly precessing or rotating with mean periods of \( T_P = 5.99685 \text{ years} \) and \( T_N = 18.5996 \text{ years} \) respectively. The (ascending) node is the direction where this plane intersects the ecliptic, and it moves as

\[
N(t) = N_0 - \frac{360^\circ}{T_N} (t - t_0) - A_n
\]

where \( N_0 = N(t_0) \) is the node longitude at the reference date and \( A_n = 0.16^\circ \sin M_\odot \). Note that the Moon crosses the Sun with a period \( t_s \), the nodes with a period \( t_n \), and the perigee with period \( t_p \), where

\[
\text{“synodic” month} \quad \frac{1}{t_s} = \frac{1}{27.321662 \text{ days}} - \frac{1}{365.25636 \text{ days}} = \frac{1}{27.53500 \text{ days}}
\]
\[
\text{“draconic” month} \quad \frac{1}{t_n} = \frac{1}{27.321662 \text{ days}} + \frac{1}{137.25636 \text{ days}} - \frac{1}{27.53500 \text{ days}} = \frac{1}{27.53500 \text{ days}}
\]
\[
\text{“anomalistic” month} \quad \frac{1}{t_p} = \frac{1}{27.321662 \text{ days}} - \frac{1}{137.25636 \text{ days}} - \frac{1}{27.53500 \text{ days}} = \frac{1}{27.53500 \text{ days}}
\]
Reference data for Moon 2000.0 (from ssd.jpl.nasa.gov/?sat_elem):

\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
a & e & $\omega_p$ & $t_0$ & i & $N_0$ & diameter \\
\hline
384399km & 0.0549006 & 83.1862° & 2000.05177 & 5.1454° & 125.1240° & 3474km \\
\hline
+40.6098°/year & & & -19.341°/year & & & \\
\hline
\end{tabular}

(It’s best to look up $t_0$ in a table of lunar perigees, e.g. \url{www.fourmilab.ch/earthview/pacalc.html}.)

One can only find the main periodic terms of this motion. A fair approximation is elliptical motion with various corrections:

Mean anomaly $M(t) = 360°(t - t_0)/27.321582$ days

‘True’ anomaly $\theta = \nu$

\begin{align*}
&+1.2739° \sin(2D - M) \quad \text{“evection”} \\
&+0.658° \sin 2D \quad \text{“variation”} \\
&-0.1858° \sin M\odot \quad \text{“annual equation”}
\end{align*}

where $D = M + \omega_p - \lambda\odot$.

Then the geocentric ecliptic longitude and latitude are given by

\begin{align*}
\lambda &= \theta + \omega', \\
\beta &= i \sin F + 0.173° \sin(2D - F),
\end{align*}

where $F = \lambda - N$.

The angle Earth-Moon-Sun is then $\theta = 180° - \lambda\odot + \lambda$, so its phase is $F = \frac{1}{2}(1 + \cos \theta)$. The distance from the Earth is approximately

$$\rho \approx \frac{1 - e^2}{1 + e \cos \theta}.$$  

If one wants to track the Moon’s motion over short periods of time, one can calculate the rates of change of $\lambda_m$ and $\beta_m$, by differentiating:

\begin{align*}
\dot{\lambda}_m &\approx \dot{\nu} = \beta(1 + e \cos \nu)^2 \approx \frac{2}{3}(1 - e^2)^{-3/2}(1 + 2e \cos \nu) \approx (0.55° + 0.06° \cos M)/\text{hour} \\
\dot{\beta}_m &\approx i \cos(\lambda - N)(\dot{\lambda} - \dot{N}) \approx 0.05° \cos(\lambda - N)/\text{hour}.
\end{align*}

See \url{main.chemistry.unina.it/~alvitagl/solex/} for an example of a direct numerical orbit calculator.

### 3.3.1 Easter Date

Easter occurs on the first Sunday after the first full moon when spring starts (for computational purposes taken to be 21st March). There is an official algorithm to calculate it, the ‘computus’. As the week, the month, and the year are mutually irrational, and as decimal representations were not yet invented when it was devised, its workings may appear as numerical wizardry based on three numbers:
1. *Littera Dominicalis*: If the letters *A*...*G* are assigned to all dates of the year, starting 1 January, the *Dominical letter* is that letter which corresponds to the Sundays (on leap years, there are two Dominical letters, but as Easter occurs after the leap day, only the second letter matters). According to the Gregorian calendar, it can be calculated by first dividing the year as \( y = 100a + b \), then finding the number of leap years \( b = 4c + d \) and the number of leap centuries \( a = 4e + f \). As \( 365 = 1 \mod 7 \), each normal year decreases the letter by one, four years by \(-5 = 2 \mod 7\), a hundred years by \( 25 \times 2 + 1 = 2 \mod 7 \), and four hundred years by \( 4 \times 2 - 1 = 0 \mod 7 \). Combined in one formula, the (second) Dominical letter is then

\[
\ell := -d + 2c + 2f \mod 7, \text{ where } A = 0, B = 1, \ldots, G = 6.
\]

Note that \( 7 - \ell \) is the weekday of 1 January, where 0=Sunday, ..., 6=Saturday. To calculate the day of the week of any given date, note that the first of a month is

<table>
<thead>
<tr>
<th>Month</th>
<th>( \ell )</th>
</tr>
</thead>
<tbody>
<tr>
<td>March, November</td>
<td>3 - \ell</td>
</tr>
<tr>
<td>April, July</td>
<td>6 - \ell</td>
</tr>
<tr>
<td>May, January</td>
<td>1 - \ell</td>
</tr>
<tr>
<td>June, February</td>
<td>4 - \ell</td>
</tr>
<tr>
<td>August</td>
<td>2 - \ell</td>
</tr>
<tr>
<td>September, December</td>
<td>5 - \ell</td>
</tr>
<tr>
<td>October</td>
<td>7 - \ell</td>
</tr>
</tbody>
</table>

(January and February refer to the following year, because a leap day may be added at the end of February.)

2. *Numerus Aureus*: The lunar synodic period is the month=29.530588 days; the approximation used in the computus is based on another fraction:

\[
\frac{29.530588}{365.25} \approx 1/(12 + 1/(2 + 1/(1 + 1/(2 + 1/2)))) = \frac{19}{235},
\]

i.e., 19 civil years are almost exactly 235 lunar months. So only 19 different Easter full-moon dates need to be calculated, one for each *golden number* \( n = 1 + (Y \mod 19) \).

3. *Epactae*: The age of the Moon on New Year’s Day (age=0=30 for a new moon, age=14 for a full moon). With each year a full moon occurs \( 13 \times 29.53 - 365 = 18.898 \) days later, or equivalently 10.63 days earlier (11.63 when a leap year); on average, it is 10.88 days/year. In the computus, the epact increases by 11 days, modulo 30, but one must take into effect the leap centuries; it would equal \( 11n - 3 + e - a \mod 30 \). This agrees with increases of 10.88 modulo 29.53, but after 19 years there remains an error of 0.061839 days, i.e., 0.00325 days/year, or approximately 8 days every 25 centuries, so a correction of \( 8a/25 \), rounded to the nearest integer, is applied

\[
\text{epact } e = 8 + 11n - a + e + \text{round}(8a/25) \mod 30.
\]
Consider 21 March:

\[
\text{day of the week} = 31 + 28 + 21 - \ell \mod 7 \\
= 3 - \ell \mod 7,
\]

\[
\text{age of Moon} = \epsilon + 31 + 28 + 21 - 30 - 29 \mod 30 \\
= \epsilon + 21 \mod 30
\]

The number of days left after 21 March until the Easter Full Moon is then

\[
h := 14 - (\epsilon + 21) \mod 30 \\
= 23 - \epsilon \mod 30
\]

The day of the week of Easter Full Moon is then \( h + 3 - \ell \mod 7 \), and the number of days left until the following Sunday is

\[
\lambda := 7 - (h + 3 - \ell) \mod 7 \\
= \ell - h + 4 \mod 7
\]

Easter should then be \( h + \lambda \) days after 22 March. However, the way the traditional computus works, the period of the moon is increased alternately by 30 and 29 days, with the ‘missing’ day in the ‘short’ months taken as the 6th day; after 11 years, it can happen, when the epact is 24 or 25 (so the New Moon is on lunar day 5/6), that two years have the same full moon date; so when \( \epsilon = 25 \) or 26 and \( n > 11 \), there is a correction of \(-1\); equivalently

\[
\mu := \begin{cases} 
1 & 11h + n \geq 320 \\
0 & \text{o/w}
\end{cases}
\]

This will only have an effect on the Easter date if \( \lambda = 6 \), in which case we need to take the previous Sunday, i.e.,

\[
\text{days from 22 March to Easter} = e := h + (\lambda + \mu \mod 7) - \mu
\]

The last step is to convert this to a date

\[
\text{month} = \begin{cases} 
3 & e \leq 9 \\
4 & e > 9
\end{cases} \\
\text{day} = \begin{cases} 
22 + e & e \leq 9 \\
1 + e - 9 & e > 9
\end{cases}
\]

or equivalently, \( e + 114 = 31 \text{month} + (\text{day} - 1) \).

There is a short algorithm, originally by Gauss and amended anonymously, that inputs the year \( Y \) and outputs the Easter date:
3.3 Lunar Orbit

Quotient Remainder

\[
\begin{align*}
Y & \div 19 & - & m \quad n = m + 1 \\
Y & \div 100 & a & b \\
b & \div 4 & c & d \\
a & \div 4 & e & f \\
8a+13 & \div 25 & g & - \\
19m+a-e-g+15 & \div 30 & h & (19 = -11 \mod 30) \\
11h+m & \div 319 & \mu & - \\
2c+2f-d & \div 7 & \ell & \text{Dominical letter} \\
h+\lambda-\mu+32 & \div 25 & j & - \\
h+\lambda-\mu+j+19 & \div 32 & i & - \\
\end{align*}
\]

The Easter date is then \(i/j/Y\). For the Easter full moon take \(\lambda = -1\).

3.3.2 Eclipses

Eclipses occur when the Moon and Sun are in exactly the same or opposite directions; since the lunar orbital plane is different from the ecliptic, these directions must be the nodes. Both the Sun and the Moon move along approximately uniformly during the eclipse,

\[
\begin{align*}
\lambda_\odot(t) &= \lambda_\odot(t_0) + \dot{\lambda}_\odot(t-t_0), \\
\lambda_m(t) &= \lambda_m(t_0) + \dot{\lambda}_m(t-t_0), \\
\beta_\odot(t) &= 0 \\
\beta_m(t) &= \beta_m(t_0) + \dot{\beta}_m(t-t_0)
\end{align*}
\]

where \(\dot{\lambda}_\odot \approx 360^\circ/\text{year} = 0.04^\circ/\text{hour}\). The Moon’s position relative to the Sun is \((\lambda_m - \lambda_\odot, \beta_m)\), a straight line with a slope of \(\dot{\beta}_m/(\dot{\lambda}_m - \dot{\lambda}_\odot)\) coming in from right to left. It can be calculated at two times to determine the straight line. The Sun’s angular diameter can be calculated from its true diameter divided by the distance Earth-Sun; similarly the Moon’s. In either case, the time of eclipse maximum is given at the point when this relative line is closest to the origin,

\[
\frac{\dot{\beta}_m + \frac{\lambda_m - \lambda_\odot}{\dot{\lambda}_m - \dot{\lambda}_\odot}}{\dot{\beta}_m} = 0 \Rightarrow t = t_0 - \frac{1}{2} \left( \frac{\beta_m(t_0)}{\dot{\beta}_m} + \frac{\lambda_m(t_0) - \lambda_\odot(t_0)}{\dot{\lambda}_m - \dot{\lambda}_\odot} \right)
\]

A solar eclipse occurs when both the Sun and the Moon are at the same node, \(\lambda_m = \lambda_\odot\) and \(|F| < 18^\circ31'\) or \(|F - 180^\circ| < 18^\circ31'\). The longest duration of a total eclipse is 7m31s, that of an annular eclipse is 12m24s.

A lunar eclipse occurs when the Sun is at one node, and the Moon at the other node; the angular size of the umbra and penumbra are, on average, 1.37° and 2.44°; more precisely, when the Earth is at distances \(a, d\) from the Sun and Moon, they are (in radians):

\[
\begin{align*}
\text{umbra} &= \frac{2R_E}{d} \left( 1 + \frac{a}{(a-1)d-1} \right)^{-1}, \\
\text{penumbra} &= 2 \left( \frac{1}{d} + \frac{1 + \alpha}{a} \right) R_E,
\end{align*}
\]

where \(\alpha = 695842/6371 = 109.22\). The longest duration of a lunar eclipse is 1h47m (3h45m from tip to tip).
Saros Cycle: The lunar eclipses are fairly predictable (unlike total solar eclipses) because of the large size of the Earth’s shadow relative to the Moon. Two numbers are important: the month relative to the sun (29.53059 days) and the time taken for the Moon to cross the orbital nodes (27.21222 days). Using continued fractions, their ratio can be approximated by

\[
\frac{m}{n} \approx 27.5m - 29.5m/27.5 - 29.5m/365.25 \text{(yr)}
\]

Since the Moon moves at a rate of \(360^\circ/29.53059\text{days} = 1.37^\circ/0.11\text{days}\), the required accuracy is almost achieved by the ratio 47/51 and better by 223/242. The former is a period of 3.8 years of recurring eclipses, typically the 0,6,12,18,24,30,36,41,42,(47) ‘moons’; but this pattern changes slightly for the subsequent 3.8 years. The second fraction leads to the Saros cycle of 18.03 years = 18 years 11 days, in which recurrent eclipses are much more similar to each other. Coincidentally, the Moon’s cycle of distances from the Earth is of 27.55455 days (third column), which almost exactly divides 223 synodic months, so corresponding eclipses in consequent Saros cycles are very similar to each other. The current Saros cycle has the following eclipses:

<table>
<thead>
<tr>
<th>Year</th>
<th>Lunar Eclipses</th>
<th>Solar Eclipses</th>
</tr>
</thead>
<tbody>
<tr>
<td>2006</td>
<td>14 Mar, 7 Sep</td>
<td>29 Mar, 22 Sep</td>
</tr>
<tr>
<td>2007</td>
<td>3 Mar, 28 Aug</td>
<td>(19 Mar), (11 Sep)</td>
</tr>
<tr>
<td>2008</td>
<td>21 Feb, 16 Aug</td>
<td>(7 Feb), (1 Aug)</td>
</tr>
<tr>
<td>2009</td>
<td>9 Feb, 7 Jul, 6 Aug, 31 Dec</td>
<td>26 Jan, 22 Jul</td>
</tr>
<tr>
<td>2010</td>
<td>26 Jun, 21 Dec</td>
<td>15 Jan, 11 Jul</td>
</tr>
<tr>
<td>2011</td>
<td>15 Jun, 10 Dec</td>
<td>(4 Jan), (25 Nov)</td>
</tr>
<tr>
<td>2012</td>
<td>4 Jun, 28 Nov</td>
<td>20 May, 13 Nov</td>
</tr>
<tr>
<td>2013</td>
<td>25 Apr, 25 May, 18 Oct</td>
<td>10 May, 3 Nov</td>
</tr>
<tr>
<td>2014</td>
<td>15 Apr, 8 Oct</td>
<td>(29 Apr), (23 Oct)</td>
</tr>
<tr>
<td>2015</td>
<td>4 Apr, 28 Sep</td>
<td>(20 Mar), (13 Sep)</td>
</tr>
<tr>
<td>2016</td>
<td>23 Mar, 18 Aug, 16 Sep</td>
<td>9 Mar, 1 Sep</td>
</tr>
<tr>
<td>2017</td>
<td>11 Feb, 7 Aug</td>
<td>26 Feb, 21 Aug</td>
</tr>
<tr>
<td>2018</td>
<td>31 Jan, 27 Jul</td>
<td>(15 Feb), (11 Aug)</td>
</tr>
<tr>
<td>2019</td>
<td>21 Jan, 16 Jul</td>
<td>2 Jul, 26 Dec</td>
</tr>
<tr>
<td>2020</td>
<td>10 Jan, 5 Jun, 5 Jul, 30 Nov</td>
<td>21 Jun, 14 Dec</td>
</tr>
<tr>
<td>2021</td>
<td>26 May, 19 Nov</td>
<td>(10 Jun), (4 Dec)</td>
</tr>
<tr>
<td>2022</td>
<td>16 May, 8 Nov</td>
<td>(30 Apr), (25 Oct)</td>
</tr>
<tr>
<td>2023</td>
<td>5 May, 28 Oct</td>
<td>20 Apr, 14 Oct</td>
</tr>
<tr>
<td>2024</td>
<td>25 Mar, 18 Sep</td>
<td>8 Apr, 2 Oct</td>
</tr>
</tbody>
</table>
4 Luminosity

The luminosity \( L \) is the emitted (visual) power output of an object. The **absolute magnitude** is a logarithmic scale of the luminosity, with respect to a reference luminosity \( L_0 := 3.31 \times 10^{28} \text{W} \),

\[
M_v := -\frac{5}{2} \log_{10} \frac{L}{L_0}
\]

The Sun has a luminosity of \( L_{\odot} = 3.8 \times 10^{26} \text{W} \), so its absolute magnitude is 4.85, and relative to it,

\[
M_v = 4.85 - \frac{5}{2} \log_{10} \frac{L}{L_{\odot}}
\]

Note that a small hot star may give out as much light as a large ‘cool’ star.

The **apparent luminosity** \( L' \) is the received light intensity after it has traveled a distance \( r \), i.e., \( L' = L/4\pi r^2 \). Since relative comparisons are much easier to do, it was first estimated as a ratio \( L'/L'_{\text{Vega}} \) with respect to the star Vega’s apparent luminosity, \( L'_{\text{Vega}} = 2.75 \times 10^{-8} \text{W/m}^2 \). The **apparent magnitude** is a logarithmic scale of the intensity

\[
m_v := -\frac{5}{2} \log_{10} \frac{L'}{L'_{\text{Vega}}}
\]

\[
= -18.9 - \frac{5}{2} \log_{10} \frac{L}{4\pi r^2} \quad \text{S.I. units}
\]

\[
= -26.74 - \frac{5}{2} \log_{10} \frac{L/L_{\odot}}{r^2} \quad r \text{ measured in a.u.}
\]

\[
= -10.32 + M_v + \frac{5}{2} \log_{10} 4\pi r^2 \quad r \text{ measured in light-years}
\]

\[
= M_v + 5 \log_{10}(r/10) \quad r \text{ measured in parsecs}
\]

For example,

<table>
<thead>
<tr>
<th>Light intensity(mW/m²)</th>
<th>Apparent magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun 1.38kW/m²</td>
<td>−26.74</td>
</tr>
<tr>
<td>Full Moon 3.31mW/m²</td>
<td>−12.7</td>
</tr>
</tbody>
</table>

For light reflected from the Sun by an object at a distance \( r \) from the Sun and \( \rho \) from the Earth, the brightness is approximately

\[
\frac{L_{\odot}}{4\pi r^2} \times \frac{\pi D^2}{4} \times F \times A \times \frac{1}{4\pi \rho^2},
\]

where \( D \) is the diameter of the object, \( F \) its phase, and \( A \) its ‘albedo’ (the fraction of light reflected); so the apparent magnitude is

\[
m_v = -26.74 + \frac{5}{2} \log_{10} \frac{16\pi^2 \rho^2}{D^2 FA} \quad r, \rho, D \text{ in a.u.}
\]

\[
= -23.7 + 5 \log_{10} \frac{r \rho}{D \sqrt{FA}}
\]

\[
= 17.2 + 5 \log_{10} \frac{r \rho}{D \sqrt{FA}} \quad r, \rho \text{ in a.u., } D \text{ in km}
\]
For a comet, the size of the coma depends on the distance from the Sun, i.e., $D$ is proportional to $r^{-1}$ (or $r^{-2}$), so $m_v = 17.2 + c + 5 \log_{10} \frac{r^2 \rho}{\sqrt{F_A}}$.

References:
- astrowww.phys.uvic.ca/~tatum/celmechs.html
- www.stjarnhimlen.se/comp/ppcomp.html
- www.moshier.net
- aa.usno.navy.mil