1 Objects and Morphisms

A category is a class of objects $A$ with morphisms $f : A \rightarrow B$ (a way of comparing/substituting/mapping/processing $A$ to $B$) such that,

(i) given morphisms $f : A \rightarrow B$, $g : B \rightarrow C$, $gf : A \rightarrow C$ is also a morphism,

(ii) for compatible morphisms, $h(gf) = (hg)f$, and

(iii) each object $A$ has a morphism $a : A \rightarrow A$ satisfying $af = f$, $ga = g$.

(Note: in a sense, an object $A$ is the morphism $a$; so we can even do away with objects.)

Sets can be considered as 0-categories (only objects or elements), or as discrete categories with each object $A$ having one morphism $a$.

The class of morphisms from $A$ to $B$ is denoted $\text{Hom}(A, B)$; thus $\text{Hom}(A, A)$ is a monoid.

Even at this abstract level there are at least three important categories:

1. logic (with statements as objects and $\Rightarrow$ as morphisms),
2. sets (with functions as morphisms),
3. computing (with data types and algorithms).

1.1 Morphisms

A monomorphism $f : A \rightarrow B$ satisfies

$$\forall C, \forall x, y \in \text{Hom}(C, A), \quad fx = fy \Rightarrow x = y.$$ 

$$C \xrightarrow{x} A \xrightarrow{f} B$$

An epimorphism $f : A \rightarrow B$ satisfies

$$\forall C, \forall x, y \in \text{Hom}(B, C), \quad xf = yf \Rightarrow x = y.$$ 

$$A \xrightarrow{f} B \xrightarrow{x} C$$

1. In particular, for a monomorphism $f$, $fg = f \Rightarrow g = \iota_A$; for an epimorphism $gf = f \Rightarrow g = \iota_B$. 

2. The composition of monomorphisms is a monomorphism, and of epimorphisms an epimorphism.

3. Conversely, if \( fg \) is a monomorphism then so is \( g \), and if it is an epimorphism then so is \( f \).

A monomorphism \( f : A \to B \) is also called a sub-object of \( B \). Monomorphisms with the same codomain have a pre-order: let \( f \leq g \) for \( f : A \to C \), \( g : B \to C \) when \( f = gh \) for some (mono)morphism \( h : A \to B \);

\[
\begin{array}{c}
A \\
\downarrow^f \\
B \\
\downarrow^g \\
C
\end{array}
\]

It can be made into a poset by using the equivalence relation \( f \cong g \) when \( f \leq g \leq f \).

An isomorphism is an invertible morphism, i.e., \( f \) has an inverse \( g \) such that \( fg = \iota_B, gf = \iota_A \). In this case, \( A \) and \( B \) are called isomorphic (an equivalence relation); iff \( f \leq g \leq f \). An isomorphism \( f : A \to A \) is called an automorphism; for example, any \( \iota_A \); the automorphisms of \( A \) form a group.

If \( gf = \iota \) then \( f \) is called a split monomorphism or section (has a left-inverse), and \( g \) a split epimorphism or retraction (has a right-inverse). A morphism with left and right inverses is an isomorphism (since then, \( g_1 = g_1 f g_2 = g_2 \)).

An extremal monomorphism is a monomorphism \( f \) such that the only way \( f = ge \) with \( e \) an epimorphism is that \( e \) is an isomorphism (and \( g \) a monomorphism). An extremal epimorphism is an epimorphism \( f \) such that \( f = eg \) with \( e \) a monomorphism \( \Rightarrow e \) is an isomorphism (and \( g \) an epimorphism). Thus a monomorphism which is an extremal epimorphism, or an epimorphism which is an extremal monomorphism, is an isomorphism.

Let \( f \perp g \) mean \( gx = yf \Rightarrow \exists u x = uf, y = gu \). A strong monomorphism is one such that \( \text{Epi} \perp f \).

Isomorphisms \( \subseteq \text{SplitMono} \subseteq \text{StrongMono} \subseteq \text{ExtremalMono} \subseteq \text{Monomorphisms} \)

Isomorphisms \( \subseteq \text{SplitEpi} \subseteq \text{StrongEpi} \subseteq \text{ExtremalEpi} \subseteq \text{Epimorphisms} \)

Proof. If \( f \) is a split monomorphism with \( gf = \iota \), then \( f \) is a monomorphism and \( g \) an epimorphism. If \( f = hk \) with \( k \) an epimorphism, then \( ghk = \iota \) and \( kghk = k \), so \( kgh = \iota \); thus \( k \) has the inverse \( gh \).

If \( f = ge \) is a strong monomorphism and \( e \) epi, then \( e \perp f \), so \( f \iota = ge \Rightarrow \exists u \iota = ue, g = fu \). So \( e \) is split and an epi, hence an isomorphism.

A morphism \( f : A \to A \) is called idempotent when \( f^2 = f \); for example, the split idempotents \( f = gh \) where \( hg = \iota \).

An object is called finite, when every monomorphism \( f : A \to A \) is an automorphism. In particular, if \( B \subseteq A \cong B \) then \( A = B \).

Example: For Sets, a monomorphism is a 1-1 function; an epimorphism is an onto function; such functions are automatically split; an isomorphism is thus a
bijective function; isomorphic sets are those with the same number of elements; a set is finite in the category sense when it is finite in the set sense.

**Functors** (or actions) are maps between categories that preserve the morphisms (and so the objects),

\[ Ff : FA \rightarrow FB, \quad F_{FA} = \iota_{FA}, \quad F(fg) = FfFg \]

They preserve isomorphisms.

### 1.2 Constructions

**Subcategory**: a subset of the objects and morphisms; a full subcategory is a subset of the objects, with all the corresponding morphisms.

**Dual category**: \( C' \) has the same objects but with reversed morphisms \( f^\top : B \rightarrow A, \) and \( g^\top f^\top := (fg)^\top; \) so \( C'' = C. \) Every concept in a theorem has a co-concept in its dual (eg monomorphisms correspond to epimorphisms); every theorem in a category has a dual theorem in the dual category. A functor between dual categories is called a dual functor; a functor from a dual category to a category is called contra-variant, \( F(fg) = F(g)f(f). \)

A dagger category is one for which there is a functor \( \dagger : C \rightarrow C', \) where

\[ (fg)^\dagger = g^\dagger f^\dagger, f^{\dagger\dagger} = f. \]

(Set cannot be made into a dagger category because there is a morphism \( \emptyset \rightarrow 1 \) but not vice-versa).

**Product of Categories**: \( C \times D \) the objects are pairs \((X,Y)\) with \( X \in C \) and \( Y \in D, \) and the morphisms are \((f,g),\) where

\[ (f_1, g_1)(f_2, g_2) := (f_1f_2, g_1g_2), \quad \iota_{(X,Y)} = (\iota_X, \iota_Y). \]

The projection functors are \( C \times D \rightarrow C, (f,g) \mapsto f, \) and \( C \times D \rightarrow D, (f,g) \mapsto g. \)

\[ (C \times D)' \sim C' \times D' \]

(The product is the categorical product in Category)

**Quotient Category**: given a category and an equivalence relation on morphisms (of same objects) \( \sim, \) then \( C/ \sim \) is that category with the same objects and with equivalence classes of morphisms. The map \( C \rightarrow C/ \sim \) defined by \( F : A \mapsto A, f \mapsto [f], \) is a functor.

**Arrow Category**: \( C\rightarrow \) consists of the morphisms of \( C \) (as objects), with the morphisms \( f \rightarrow g \) being pairs of morphisms \((h,k)\), such that \( kf = gh, \)

\[ \begin{array}{ccc}
    f & \downarrow & g \\
    h & \rightarrow & k
  \end{array} \]

and composition \((h_1, k_1)(h_2, k_2) := (h_1h_2, k_1k_2),\) and identities \((\iota_A, \iota_B).\) Monomorphisms are those pairs \((h,k)\) where \( h \) and \( k \) are monomorphisms. For example, the arrow category of sets is the category of functions.
Slice Category (or comma category): $C \downarrow B$ is the subcategory where the morphisms have the same codomain $B$ and $k = \iota$; the morphisms simplify to $h$ where $f = gh$; similarly for the morphisms with the same domain. An object $A$ is called projective when every morphism $f : A \to B$ factors through any epimorphism $g : C \to B$, $f = gh$. Dually, $A$ is called injective when $f : B \to A$ factors through any monomorphism $f = hg$.

1.3 Functors

Functors can be thought of as higher-morphisms acting on objects and morphisms; or as a model of $C$ in $D$.

(Examples: the constant functor, mapping objects to a single one, and morphisms to its identity; the mapping from a subcategory to the parent category; forgetful functor (when structure is lost) and inclusion functor (when structure is added, minimally); the mapping which sends $A$ to the set $\text{Hom}(B, A)$ and a morphism $f$ to the function $g \mapsto f \circ g$ is a functor from any category to the category of sets; similarly for $A \mapsto \text{Hom}(A, B)$ and $f \mapsto (g \mapsto g \circ f)$ (contra-variant).)

A functor is called faithful when it is 1-1 on morphisms (and hence objects) It is full when it is onto all morphisms in $\text{Hom}(FA, FB)$; it is called dense, when it is onto all objects up to isomorphism. It is an isomorphism on categories when it is bijective on the morphisms $\text{Hom}(FA, FB)$. A dense isomorphism is called an equivalence, and the two categories are said to be equivalent $A \sim B$.

A (left) adjoint of a functor is $F^* : D \to C$ with natural isomorphisms $e, i$ such that $e : FF^* \to 1, i : 1 \to F^*F$ and $\text{Hom}(F^*A, B) \sim \text{Hom}(A, FB)$; hence $(FG)^* = G^*F^*$. (For example, a forgetful functor and inclusion functor are adjoints, with $i$ being the embedding)

2-Categories: Categories with functors as morphisms form a Category; the identity functor is the one which leaves objects and morphisms untouched; (there is an initial object namely $\emptyset$, and a terminal object, $\{ \}$. It has the additional structure of a 2-functor, called a “natural transformation” (or ‘homotopy’), between functors on the same categories, $\tau : F \to G$; two such functors map an object $A \in C$ to two objects $FA$ and $GA$ in $D$, and a natural transformation determines a morphism $\tau_A : FA \to GA$ between the two, such that $\forall f : A \to B, (Gf)\tau_A = \tau_B(Ff)$ (so $Ff \sim Gf$). A natural isomorphism is a natural transformation for which $\tau_A$ are isomorphisms.

With these notions, two categories are equivalent when there are functors $F$ and $F^*$ such that $F^*F \sim 1, FF^* \sim 1$ (or equivalently when $F$ and $F^*$ are isomorphisms with $F^*F \sim 1$). The auto-equivalences of a category form a symmetric monoidal category.

More generally, a 2-category is a set of objects $A$, with morphisms $f : A \to B$, and 2-morphisms $\tau : f_1 \to f_2$ (for some $f_1, f_2 \in \text{Hom}(A, B)$); 2-morphisms can be combined either “vertically” by composition $\tau_2 \tau_1$, (and must be associative,
with an identity), or “horizontally” \( \sigma \circ \tau : \sigma(g) \tau(f) \), such that

\[
\begin{array}{c}
\tau_1 \downarrow \\
g \downarrow \\
\tau_2 \downarrow \\
\end{array}
\begin{array}{c}
\tau_1 \downarrow \\
\tau_2 \downarrow \\
\end{array}
\begin{array}{c}
\sigma_1 \downarrow \\
\sigma_2 \downarrow \\
\end{array}
\begin{array}{c}
\tau_2 \tau_1 \circ \sigma_2 \sigma_1 = (\sigma_2 \circ \tau_2)(\sigma_1 \circ \tau_1).
\end{array}
\]

A 2-category with 1 object gives rise to a monoidal category (of the morphisms and 2-morphisms of the object); a 2-category with 1 object and 1 morphism gives a commutative monoid (of 2-morphisms).

The functors themselves form a category \( \mathcal{D}^C \) where morphisms are the natural transformations. \( C^1 \sim C; C^2 \) is the category of arrows on \( C \).

### 2 Limits

When a category maps under a functor \( F : C \to D \) to another category, the image of an object may have morphisms that were not present in \( C \); an object \( A \in D \) may sometimes determine a unique (up to isomorphism) object (called a universal) \( U_A \) in \( C \), which makes \( F(U_A) \) closest to \( A \) in the sense that there is a unique morphism \( \phi_A : F(U_A) \to A \), such that

\[
\forall f : F(B) \to A, \exists! g : B \to U_A, f = \phi_A F(g).
\]

\( A \) co-universal is similarly an object \( U_A \in C \) with a morphism \( \phi_A : A \to F(U_A) \) such that \( \forall f : A \to F(B), \exists! g : U_A \to B, f = F(g) \phi_A \).

In particular, sub-categories \( C \) may have universal properties:

- **Terminal object 1:** \( \forall A, \exists! f : A \to 1 \) (for the empty sub-category). **Initial object 0:** \( \forall A, \exists! f : 0 \to A \).

\((0, 0)\) is an initial object in \( C \times D \). For example, \( \{0\} \) and \( \emptyset \) are the terminal and initial objects of sets; \( \text{TRUE} \) and \( \text{FALSE} \) are the ones for logic.

**Isomorphism** The closest objects for an object \( A \) with its identity morphism (the category 1), are its isomorphic copies. For example, sets with the same cardinality are isomorphic, while statements \( A \Leftrightarrow B \) are so in logic.

**Products:** For the subcategory 2 (with only the identity morphisms), the closest object of \( A \) and \( B \) is \( A \times B \), with morphisms \( \pi_A : A \times B \to A, \pi_B : \)
$A \times B \to B$ such that any other morphisms $p_A : C \to A$, $p_B : C \to B$ factor out through a unique morphism $g : C \to A \times B$, $p_A = \pi_A g$, $p_B = \pi_B g$.

\[ \begin{array}{c}
A \\
|\quad| \\
|\quad| \\
|\quad| \\
C \xrightarrow{g} A \times B \\
|\quad| \\
|\quad| \\
|\quad| \\
B \\
\end{array} \]

$1 \times A \cong A; A \times B \cong B \times A; (A \times B) \times C \cong A \times (B \times C)$.

For example, the usual product $A \times B$, and the statement $A$ and $B$ are the products for sets and logic respectively.

More generally, starting with a discrete category, the closest object of $A_i$ is $\prod_i A_i$, with $\pi_i : \prod_i A_i \to A_i$ i.e., if $p_i : X \to A_i$ are morphisms then there is a morphism $h : X \to \prod_i A_i$ with $p_i = \pi_i h$. A repeated product gives $A \times C$ (starting with a constant functor from a discrete category).

A relation on objects $A, B$ is a monomorphism $R : \rho \to A \times B$.

Sums (or Co-products): $\coprod_i A_i$ is the dual of the product in the dual category i.e., it is the closest object with morphisms $\pi_i : A_i \to \coprod_i A_i$. For example, $A + B$ (disjoint union) and $A$ or $B$.

Equalizer: starting from the category with two objects $A$, $B$, and morphisms $f_i : A \to B$, their equalizer is the closest object $E$ with (extremal mono-)morphism

\[ eq : E \to A, \quad \forall i, j, f_i eq = f_j eq. \]

\[ E \xrightarrow{eq} A \xleftarrow{f_1 \ f_2} B \]

For example, for Sets, $\{ x : f_1(x) = f_2(x) \}$.

Equalizers are monomorphisms: let $e = eq$, if $xe = ye$ then $xef = xeg$, so $\exists u, xeue = xu = x$; similarly $y = u = x$.

Co-equalizer: similarly an (extremal epi-)morphism

\[ coeq : Y \to E, \quad \forall i, j, coeqf_i = coeqf_j. \]

For example, the co-equalizer of a relation on a set $X$ is the partition on it (for an equivalence relation, this partition is compatible with the relation).

Pullback (fibre product): starting from the category with objects $X_i$ and morphisms $f_i : X_i \to Z$, then the pullback is the (unique...) closest object $\prod_Z X_i$ with morphisms

\[ \pi_i : \prod_i X_i \to X_i, \quad f_i \pi_i = \pi_Z. \]
The equalizer is a special case when the morphisms start from the same object. If $Z$ is the terminal object, then $\prod_Z X_i = \prod_i X_i$. For example, the pullback on sets is $X \times_Z Y = \{ (x, y) : f(x) = g(y) \}$; in particular when $g$ is the identity, $X \times_Z Y = f^{-1}Y$.

Pullback lemma: pullbacks form squares $(X \times_Z Y, X, Z, Y)$; if two adjacent squares form pullbacks, then so does the outer rectangle; if the outer rectangle and the right (or bottom) square are pullbacks, then so is the left (or upper) square.

Pullbacks preserve monomorphisms: If $f \circ u = g \circ v$ with $f$ mono, and $g \circ x = g \circ y$, then $f \circ u \circ x = g \circ v \circ x = g \circ u \circ y$, so $u \circ x = u \circ y$ and $x = y$ by uniqueness of pullbacks.

**Push-out** is that closest object $\bigvee_i X_i$ with $\pi_i : X_i \to \bigvee_i X_i$, $\pi_i \circ f_i = \pi_Z$.

For example, for sets, the push-out $X \cup_Z Y$ is the set $X \cup Y$ with the elements $f(z) \in X$ and $g(z) \in Y$ identified.

**Inverse Limit**: starting from the subcategory of a chain of objects $A_i$ with morphisms $f_{j,i}$ (such that $f_{k,i} = f_{k,j} \circ f_{j,i}$), the inverse limit is the closest object $\lim_{\leftarrow} A_i$ with morphisms

$$
\pi_i : \lim_{\leftarrow} A_i \to A_i, \quad \pi_j \circ f_{j,i} = \pi_i.
$$

$$
\lim_{\leftarrow} A_i \xrightarrow{\pi_i} A_3 \xrightarrow{f_{32}} A_2 \xrightarrow{f_{21}} A_1
$$

(More generally, can start with a topology of objects rather than a chain.) The pullback is a special case. For example, the inverse limit of sets $X_i$ is the set of sequences $x_i \in X_i$ such that $x_j = f_{j,i}(x_i)$.

**Co-limit** (Direct Limit) is similar with $\lim_{\rightarrow} A_i$ and morphisms

$$
\pi_i : A_i \to \lim_{\rightarrow} A_i, \quad \pi_i \circ f_{j,i} = \pi_j.
$$

More generally, for any subcategory, or any functor, $F : C \to D$ there may be a limit object $\lim F$ in $D$ with (unique) morphisms $\pi_A : \lim F \to A$ ($A \in C$) such that for any $f : A \to B$, $A, B \in C$,

$$
\begin{align*}
    f \circ \pi_A &= \pi_B \\
    \lim F \xrightarrow{\pi_A} A \xrightarrow{f} B
\end{align*}
$$
and it is the closest such object in the sense that for any other \( C \in \mathcal{D} \) with \( f_{PA} = p_B \) then \( \exists u : C \rightarrow \lim F, \pi_A u = p_A \). A limit, if it exists, is unique up to isomorphism.

A **co-limit** is similar with \( f_A : F(A) \rightarrow \mathrm{colim}F \) such that

\[
\forall f : A \rightarrow B, \quad f_B F(f) = f_A.
\]

In general, any functor from a category with an initial object to \( \mathcal{C} \) has a limit; and any functor from a category with a terminal object has a co-limit.

A **complete** category is one in which every subcategory (or functor) has a limit. For example, the category of sets is complete and co-complete.

A **pre-sheaf** is a contra-variant functor from a pre-order (or topology) to a category \( F : \mathcal{O} \rightarrow \mathcal{C} \) (the \( F(x) \) are called sections of \( F \) over \( x \)) such that \( x \leq y \Rightarrow \) there is a restriction morphism \( F(x) \rightarrow F(y) \) with \( \mathrm{res}_{x,y} = \iota_{F(x)} \) and \( x \leq y \leq z \Rightarrow \mathrm{res}_{y,z} \\mathrm{res}_{z,y} = \mathrm{res}_{z,x} \).

A **sheaf** is a continuous pre-sheaf (preserves limits). On a topological space \( X \), the **stalk** at \( x \in X \) is the direct limit of the open neighborhoods of \( x \). So there is a morphism \( F(U) \rightarrow F_x \) for \( x \in U \) open (if the morphism is a function \( f \mapsto f_x \), where \( f_x \) is called the germ at \( x \)). The **etale** space \( E \) is the space of stalks, with the continuous map \( E \rightarrow X, F_x \rightarrow x \). (The set of sheaves form
a topos, with \( \Omega = \) the disjoint union of all open sets) The space \( E \) is locally homeomorphic to \( X \) (i.e., there are isomorphic open sets in \( E \) and \( X \) that cover \( F \) and \( x \)).

For example, a sheaf of sets is a bundle, i.e., a collection of disjoint sets \( A_i \) with a map \( \pi : \bigcup_i A_i \to I \), \( \pi^{-1}(i) = A_i \); the category of bundles over \( I \) is the same as the comma category.

### 2.1 Monoidal Categories

Objects have an associative functor tensor product \( A \otimes B \) and an object \( I \) (called unit) such that

\[
I \otimes A \cong A \cong A \otimes I \\
(A \otimes B) \otimes C \cong A \otimes (B \otimes C) \\
(A \otimes I) \otimes B \cong A \otimes B \cong A \otimes (I \otimes B)
\]

(the isomorphisms in the first two lines are called the two unitor and one associator natural isomorphisms; more generally, any product of \( n \) objects are isomorphic to each other). Product of morphisms \( f \otimes g : A \otimes B \to C \otimes D \).

The tensor product is like treating two objects in parallel; so a morphism \( f : A \otimes \ldots \otimes B \to C \otimes \ldots \otimes D \) takes \( n \) objects and “maps” them to \( m \) objects, and looks like a Feynman diagram. The unit object is null, so \( f : I \to A \) “creates” one object. The tensor product is different from the categorical product in that there need not be projections.

The morphisms \( \text{Hom}(I,I) \) now have two operations: \( (f \otimes g)(h \otimes k) = (fh) \otimes (gk) \); but from universal algebras, this implies that \( f \otimes g = fg \) and is commutative.

Set with \( \times \) is monoidal (in fact cartesian-closed); Set with disjoint union is also monoidal.

The (right) dual of an object \( A \) is another object \( A^* \) (unique up to isomorphism), such that there are “annihilation/creation” morphisms

\[
A \otimes A^* \to I, \quad I \to A^* \otimes A,
\]

called the co-unit of \( A \) and the unit of \( A \), respectively, satisfying the zig-zag equations, i.e., creating then annihilating \( A \) and \( A^* \) leaves nothing \( I \); \( A^* \) can be represented as a line in the opposite direction of \( A \); \( A \) is called the left dual of \( A^* \).

### 2.1.1 Braided Monoidal categories

A monoidal category in which there is a natural isomorphism that switches objects around,

\[
A \otimes B \cong B \otimes A,
\]
such that all permutations of products become isomorphic, e.g. \((A \otimes B) \otimes C \cong C \otimes (B \otimes A)\), i.e.,

\[
\begin{array}{ccc}
A & B \\
\downarrow & \downarrow \\
B & A
\end{array}
\quad \begin{array}{ccc}
A & C \\
\downarrow & \downarrow \\
B & A
\end{array}
\quad \begin{array}{ccc}
C & A \\
\downarrow & \downarrow \\
A & B
\end{array}
\]

It need not be its own inverse! Its inverse is:

\[
\begin{array}{ccc}
B & A \\
\uparrow & \uparrow \\
A & B
\end{array}
\]

The Yang-Baxter equation states

\[
\begin{array}{ccc}
A & C & A \\
B & \rightarrow & B \\
C & A & C \\
\rightarrow & \rightarrow & \rightarrow
\end{array}
\]

Left duals are duals.

A braided monoidal category is called \textit{symmetric} when the switching isomorphism is its own inverse.

\section{2.2 Closed Monoidal Categories}

A monoidal category is \textbf{closed} when every set of morphisms \(\text{Hom}(A, B)\) has an associated object \(B^A\), with

\[
\text{Hom}(A \otimes B, C) \cong \text{Hom}(B, C^A)
\]

(or alternatively \(\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, C^B)\)) (via “currying” natural isomorphisms). That is, every morphism can be treated as an object (without inputs). In particular \(f : A \rightarrow B\) is associated to \(I \rightarrow B^A\).

For example, in sets, the powerset axiom asserts that \(\text{Hom}(A, B)\) is a set \(B^A\); in logic the distinction is between the morphism \(A \vdash B\) and the object \(A \Rightarrow B\).

A monoidal category is \textbf{compact} (or \textit{autonomous}) when every object has a dual and a left dual. In this case it is closed, with \(A^B := B^* \otimes A\), i.e., \(A^* \cong \text{Hom}(A, I)\); in particular the unit \(I\) corresponds to a unit inside \(A^* \otimes A\).

The reverse of currying, changing an object into a morphism, is an \textit{evaluation} morphism

\[
\text{eval} : A \otimes B^A \rightarrow B, \quad \text{eval}(f \otimes \iota_A) = f.
\]
(So morphisms of two variables become morphisms of one variable.)

For example, in sets (and functional programming languages), \( \text{eval}(f, x) = f(x) \); in logic, it is modus ponens, \( A \land A \Rightarrow B \) gives \( B \).

### 2.3 Cartesian-closed categories

Finite products exist and are closed, i.e., every functor \( \times A \) has a right-adjoint \( A \), called exponentiation,

\[
\text{Hom}(A \times B, C) \cong \text{Hom}(B, C^A)
\]

This means that every morphism \( f : \prod_i A_i \to C \) can be represented by an ordered set of morphisms \( f_i : A_i \to C \).

It is thus symmetric braided monoidal, with \( \otimes \) being \( \times \) and the unit being the terminal object \( 1 \); but has more properties in that it can duplicate objects via \( \Delta : A \to A \times A \); and delete objects by mapping to \( 1 \), i.e., \( ! : A \to 1 \); every morphism \( f : 1 \to A \times B \) is of the type \( (1, 1) : 1 \to A, 1 \to B \). (e.g. the adjoint of \( X \mapsto X \times X \) is \( X \mapsto X \times Y \).)

\( f \times g : A \times B \to C \times D \) can be defined as that unique morphism induced by \( f \pi_A, g \pi_B \). In particular, \( (1_a, 1_b) = 1_{a \times b} \). Similarly, can define the sum \( f + g \).

#### 2.3.1 Evaluation

\[
\text{eval} : A \times B^A \to B, \quad \text{eval}(f \times \iota_A) = f.
\]

An element or point of \( A \) is a morphism \( x : 1 \to A \); so \( \text{eval}(f, x) = fx \).

In particular a morphism \( f : A \to B \) corresponds to an element \( 1 \to B^A \) (called the name of \( f \)).

In such categories, dual concepts lose their symmetry:

There are no morphisms \( A \to 0 \) unless \( A \cong 0 \), in particular if \( 0 \cong 1 \), then all objects are isomorphic; \( 0 \to A \) is monic.

\[
0 \times A \cong 0, \quad A^1 \cong A, \quad A^0 \cong 1, \quad 1^A \cong 1
\]

(proofs: there is only one morphism \( 0 \to B^A \), so only one morphism \( 0 \times A \to B \) so \( 0 \cong 0 \times A \), and \( A \to 0 \times A \cong 0 \to A \) forces them to be isomorphisms; \( \text{eval} : A^1 \to A \) is an isomorphism; \( 1 \to A^0 \) corresponds to \( 0 \cong 1 \times 0 \to A \) which is unique, so \( 1 \to A^0 \) and \( A^0 \to 1 \) are inverses; \( 1^A \to 1 \) must be \( \iota \) and \( 1 \to 1^A \) corresponds to \( A \to 1 \) also unique; any map \( B \to 0 \) is a unique isomorphism so \( fg = fh \Rightarrow g = h \))

\[
X^{A+B} \cong X^A \times X^B, \quad (A \times B)^C \cong A^C \times B^C,
\]
\[(C^A)^B \cong C^{A \times B}; \quad X \times (A + B) \cong X \times A + X \times B\]

(Proves: the inclusions \(A, B \rightarrow A + B\) give \(X^{A+B} \rightarrow X^A \times X^B\); conversely, \(X^A \times X^B \rightarrow X^{A+B}\) correspond to \(A + B \rightarrow X^{A \times X^B}\), i.e., to two inclusion maps, and hence the projections \(X^A \times X^B \rightarrow X^A, X^B\).

The projections \(A \times B \rightarrow A, B\) give rise to a map \((A \times B)^C \rightarrow A^C \times B^C\); its inverse is \(A^C \times B^C \rightarrow (A \times B)^C\) which corresponds to \(C \times A^C \times B^C \rightarrow A \times B\) i.e., to \(C \times A^C \times B^C \rightarrow A, B\), i.e., the projections \(A^C \times B^C \rightarrow A^C, B^C\);

\((C^A)^B \rightarrow (C^A)^B\) corresponds to \(B \times C^A \times B \rightarrow C^A\), i.e., the evaluation map \(A \times B \times C^A \times B \rightarrow X\), similarly \((C^A)^B \rightarrow C^A \times B\) corresponds to the double evaluation \(B \times A \times (C^A)^B \rightarrow C;\)

The maps \(A + B \rightarrow (X \times A + X \times B)^X\) correspond to the inclusions \(X \times A, X \times B \rightarrow X + A \times X \times B\) There is a functor mapping morphisms \(f : X_1 \rightarrow X_2\) to \(Ff : X_1^\Omega \rightarrow X_2^\Omega\) defined by \((Ff)\Omega = fg\) for \(g : Y \rightarrow X_1\). There is another contra-variant functor (restriction?) mapping morphisms \(f : Y_1 \rightarrow Y_2\) to \(Ff : Y_1^\Omega \rightarrow Y_2^\Omega\), defined by \((Ff)\Omega = gf\).

### 2.4 Topos

A category with finite limits, exponentials (i.e., cartesian-closed), and a sub-object classifier.

A **sub-object classifier** is an object \(\Omega\) (unique up to isomorphism) and a morphism \(\text{True} : 1 \rightarrow \Omega\) such that monomorphisms \(f : A \rightarrow B\) (“sub-objects”) correspond to unique morphisms

\[\chi_f : B \rightarrow \Omega, \quad \chi_f f = A \rightarrow 1 \rightarrow \Omega\]

In particular \(\text{True}\) corresponds to \(\chi_{\text{True}} = \iota_\Omega\), and the unique monomorphism \(0 \rightarrow \Omega\) corresponds to a morphism \(- : \Omega \rightarrow \Omega\); hence \(\text{False} := -\text{True} : 1 \rightarrow \Omega\).

For example, for sets \(\Omega = 2\); sub-objects \(B : I \rightarrow X\) correspond to subsets \(B \subseteq X\); subsets are maps \(A \rightarrow 2\) and correspond to the characteristic maps \(\chi_A : 1 \rightarrow 2^A\); a **singleton** is a map \(A \rightarrow 2^A\).

Other logical connectives are defined in terms of their characteristic maps:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{AND} : \Omega \times \Omega \rightarrow \Omega)</td>
<td>((\text{True, True}) : 1 \rightarrow \Omega \times \Omega)</td>
</tr>
<tr>
<td>(\text{OR} : \Omega \times \Omega \rightarrow \Omega)</td>
<td>((\text{True}<em>\Omega, \iota</em>\Omega), (\iota_\Omega, \text{True}_\Omega) : \Omega + \Omega \rightarrow \Omega \times \Omega)</td>
</tr>
<tr>
<td>(\Rightarrow : \Omega \times \Omega \rightarrow \Omega)</td>
<td>(2 \rightarrow \Omega \times \Omega\text{(where 2 is the category 0 \leq 1)})</td>
</tr>
<tr>
<td>(\sim \chi_f)</td>
<td>(\chi_f)</td>
</tr>
<tr>
<td>(\chi_f \circ g := \chi_f \text{ AND } \chi_g)</td>
<td>(\text{complement of } f )</td>
</tr>
<tr>
<td>(\chi_f \cap g := \chi_f \text{ OR } \chi_g)</td>
<td>(\text{intersections } f \cap g)</td>
</tr>
<tr>
<td>(\text{unions } f \cup g)</td>
<td>(\text{unions } f \cup g)</td>
</tr>
</tbody>
</table>
But there may be several truth values, i.e., $\Omega$ may have several elements $1 \to \Omega$, not just True and False.

$\Omega$ is injective, i.e., for any monomorphism $f : A \to B$ and any morphism $g : B \to \Omega$ there is a morphism $g : B \to \Omega$ such that $g = \hat{g}f$. $\Omega^A$ can be thought of as a “dual” of $A$; the Fourier map $\hat{\cdot} : A \to \Omega^A$ defined by $\hat{\cdot}(f) = fx$.

$f \cong g \iff \chi_f = \chi_g$; the sub-objects of $A$ form a bounded lattice, $Sub(A) \cong Hom(A, \Omega)$. A morphism is an isomorphism $\iff$ it is both mono and epi (called a bi-morphism) (since an epi monomorphism $f : A \to B$ is the equalizer of $\chi_f$ and True$_B$). Every morphism factors as $f = gh$ where $h$ is epi and $g$ is mono (via the object $fA$ obtained by the pushout of $f$ with itself). The pull-back of an epimorphism is also epi. Coproducts preserve pullbacks. (implies finite co-limits also exist)

Every category can be extended to a topos. The product of topoi is a topos. A comma category $C/A$ of a topos is also a topos; its elements are bundles of elements (i.e., sections) of $A$.

Every topos has power objects $P(A) := \Omega^A$, meaning objects $P(A)$ and $\epsilon_A$ and a monomorphism $\in : \epsilon_A \to P(A) \times A$ such that every relation (i.e., monomorphism) $r : R \to B \times A$ has an associated unique morphism $f_r : B \to P(A)$ such that $R \to B \times A \to P(A) \times A = R \to \epsilon_A \to P(A) \times A$.

$$
\begin{array}{ccc}
\epsilon_A & \longrightarrow & A \times P(A) \\
\uparrow & & \uparrow \\
R & \longrightarrow & A \times B
\end{array}
$$

$\Omega \cong P(1)$. Conversely every category with finite limits and power objects is a topos.

### 2.4.1 Well-pointed topos

A topos that satisfies the extensionality axiom, elements are epi:

$$
\forall x : 1 \to A, fx = gx \Rightarrow f = g.
$$

A morphism is mono $\iff$ it is 1-1, i.e., $fx = fy \Rightarrow x = y$ for all $x, y : 1 \to A$. A morphism is epi $\iff$ it is onto, i.e., $\forall y : 1 \to B, \exists x : 1 \to A, fx = y$.

The only non-empty object (i.e., without any elements $1 \to A$) is the initial object (since $\chi_1 \neq \chi_0$). The only elements of $\Omega$ are True and False (bivalent), and $\Omega \cong 1 + 1$ (Boolean). In fact a topos is well-pointed $\iff$ the only non-empty object is the initial one, and $\Omega \cong 1 + 1$.

The arrow category $Set^\to$ is neither Boolean nor bivalent; $Set^2$ is Boolean but not bivalent; the category of actions of a monoid (that is not a group) is bivalent but not Boolean.

### 2.4.2 With Axiom of Choice

A category is called **balanced** when $f$ is an isomorphism $\iff$ it is a monomorphism and an epimorphism.
A category satisfies an **Axiom of Choice** when every epimorphism is right-invertible (splits). So balanced.

For example, in sets, every monomorphism has a left-inverse, except for \(0 \to A\); the axiom of choice says that every epimorphism has a right-inverse.

**Strong Axiom of Choice**: \(\forall f, \exists g, f = fgf\).

A topos with the axiom of choice has the **localic** property: \(\exists i : C \to 1\) monomorphism and \(g_1 \neq g_2 \Rightarrow \exists f : C \to A, g_1f \neq g_2f\).

Also every object has a complement \(X = A + A'\).

### 2.5 Pre-additive Categories

When \(\text{Hom}(A, B)\) is an abelian group, distributive over composition of morphisms ie \(f(g + h) = fg + fh, (f + g)h = fh + gh\). (then \(\text{Hom}(A, A)\) is a ring)

Can be extended to an Abelian category.

#### 2.5.1 Additive Categories

A pre-additive category with finite products and sums;

#### 2.5.2 Abelian Categories

an additive category in which every morphism has a kernel and a co-kernel (so there is a zero object), and every monomorphism is a kernel and every epimorphism is a co-kernel.

### 2.6 Concrete category

one in which the objects are sets and the morphisms are functions; ie a category which has a faithful functor \(C \to \text{Sets}\) (called the forgetful functor).

#### 2.6.1 Category of Sets

One can even consider set theory from the categorical point of view with the following axioms:

1. Sets and functions form a category;
2. Sets have finite limits and co-limits;
3. Sets allow exponentiation;
4. Sets have a sub-object classifier (so form a topos); this is a form of comprehension axiom;
5. With a morphism \(T : 1 \to 2\);
6. Sets are Boolean in the sense that the truth-value object 2 is given by \(1 + 1\);
7. 2 has two elements (up to isomorphism);
8. Axiom of Choice (every epimorphism has a right-inverse);
9. There is an infinite (inductive) set.

It then follows that for every \( A \neq 0 \), \( \exists A \rightarrow 1 \) epimorphism and \( \exists x : 1 \rightarrow A \) morphisms (since \( A \rightarrow 1 \) is unique, which gives \( A \rightarrow B \rightarrow 1 \) where \( A \rightarrow B \) is an epimorphism; but \( A \neq 0 \Rightarrow B \neq 0 \), so \( B = 1 \); the axiom of choice gives a morphism \( x : 1 \rightarrow A \)); every monomorphism \( A \rightarrow B \) induces a “complement” monomorphism \( A' \rightarrow B \) (the pullback of \( B \rightarrow \Omega \) along \( F : 1 \rightarrow \Omega \)).

3 Research Questions

Most grand questions in pure mathematics are of the following type:

1. Syntax: given a set of mathematical structures/examples, to find a minimal set of axioms common to all.
2. Semantics: given a set of axioms, to discover all mathematical examples satisfying them; classify all possible spaces \( X \) in a category i.e., give a concrete description of the spaces, up to isomorphism.

This problem may be too hard or even impossible to answer, so the first attempt is to restrict \( X \) to the smaller ones, or else ask an easier question

2a. Find a way of distinguishing spaces: given any two spaces \( X \), \( Y \) is there a way of showing whether they are isomorphic or not?
2b. Can one show whether \( X \) is isomorphic to a known space?
2c. In particular is \( X \) isomorphic to the trivial space?