MAT1801 Mathematics for Engineers I

# Functions and Limits

## 1. Numbers

The *natural numbers* are the numbers used in counting: $1, 2, 3, \ldots$ [1]. The set of natural numbers is denoted by $\mathbb{N}$.

The *integers* (or whole numbers) are $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. The set of integers is denoted by $\mathbb{Z}$.

A *rational number* is one that can be written as a fraction (i.e. an integer divided by another integer) e.g. $\frac{1}{2}, -\frac{7}{3}, 0.305$. The set of rational numbers is denoted by $\mathbb{Q}$.

Note that any number which has a finite number of digits after the decimal point, or has a recurring decimal expansion, is rational. For example, $\frac{1}{9} = 0.11111\ldots$.

An *irrational number* is one that cannot be written as a fraction. Examples include square roots (and cube roots, fourth roots etc.) of prime numbers as well as important mathematical constant such as $\pi$ and $e$.

The set of *real numbers* includes both rational and irrational numbers, and is denoted by $\mathbb{R}$.

An *interval* is a subset of $\mathbb{R}$ that includes all the numbers between two specified real numbers. Let $a, b \in \mathbb{R}$. Then we have

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R} : a \leqslant x \leqslant b\} \\
(a, b) &= \{x \in \mathbb{R} : a < x < b\} \\
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\end{aligned}$$

The interval $[a, b]$ is said to be *closed*, while $(a, b)$ is said to be *open*. The above examples of intervals are all bounded, but we use a similar notation to denote sets which do not have both endpoints:

$$\begin{aligned}
[a, \infty) &= \{x \in \mathbb{R} : a \leqslant x\} \\
(-\infty, b) &= \{x \in \mathbb{R} : x < b\} \text{ etc.}
\end{aligned}$$

[1] Some books include 0 in the set of natural numbers.

## 2. Functions

**Definition**: A function is a rule that maps a number in a set, called the domain, to a *unique* number in another set, called the codomain.

**Examples**:

(a)
$$\begin{aligned} f_1 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

(b)
$$\begin{aligned} f_2 : A &\to \mathbb{R} \\ x &\mapsto \tan x \end{aligned}$$

where, in (b), the set $A$ is given by

$$A = \{x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2}\}.$$

Note that a function is not necessarily given by an expression. The following is a perfectly valid way of defining a function:

$$\begin{aligned} f_3 : [0, 1] &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \end{aligned}$$

An important function that we shall use very often is the *absolute value* (or *modulus*) function:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases} \end{aligned}$$

The absolute value function may be thought of a representing *distance* on the real number line. So, $|x|$ is the distance of the number $x$ from 0, while $|x - y|$ is the distance between the numbers $x$ and $y$.

## 3. Limits

**Definition**: A function $f$ is said to tend to a limit $L$ as $x$ approaches $a$ if $|f(x) - L|$ can be made as small as we wish by taking values of $x$ sufficiently close to $a$.

More formally, we say that $f$ tends to a limit $L$ as $x$ approaches $a$ if for every positive number $\epsilon$, we can find a positive number $\delta$ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

In this case, we write $\lim_{x \to a} f(x) = L$, or $f(x) \to L$ as $x \to a$.

**Theorem**: Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$. Then

1. $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + K$;
2. $\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right) = L \cdot K$;
3. for any constant $c$, $\lim_{x \to a} (c \cdot f(x)) = c \cdot \lim_{x \to a} f(x) = c \cdot L$;
4. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{K}$, provided $K \neq 0$.

Sometimes we write $\lim_{x \to a} f(x) = \infty$. What this means is that the function does **not** have a limit as $x \to a$, but that we can make $f(x)$ as large as we wish by taking values of $x$ aproaching $a$.

When the function has a horizontal asymptote, we may write $\lim_{x \to \infty} f(x) = L$. This means that we can make $|f(x) - L|$ as small as we wish by taking larger and larger values of $x$.

## 4. Continuous Functions

Roughly speaking, a continuous function is one whose graph is an unbroken line or curve. A more precise formulation of this concept is the following:

**Definition**: The function $f$ is said to be *continuous at* $a$ if

1. it is defined at $a$ and at points close to $a$;
2. $\lim_{x \to a} f(x) = f(a)$.

More formally, we say that $f$ is continuous at $a$ if for every positive number $\epsilon$, we can find a positive number $\delta$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

A function is said to be continuous if it is continuous at every point of its domain.

From the above theorem on limits, one can prove that if $f$ and $g$ are continuous, then $f+g$ and $f \cdot g$ are also continuous, and $\frac{f}{g}$ is continuous at every point where $g(x)$ is not equal to zero.

## 5. One-Sided Limits

**Definition**: $L$ is said to be the *right-hand limit* of the function $f$ at $a$ if $f$ approaches $L$ as $x$ approaches $a$ from above (i.e. through numbers larger than $a$).

The right-hand limit of $f$ at $a$ is denoted by $\lim\limits_{x \downarrow a} f(x)$ or $\lim\limits_{x \to a^+} f(x)$ or $f(a+)$.

**Definition**: $L$ is said to be the *left-hand limit* of the function $f$ at $a$ if $f$ approaches $L$ as $x$ approaches $a$ from below (i.e. through numbers smaller than $a$).

The left-hand limit of $f$ at $a$ is denoted by $\lim\limits_{x \uparrow a} f(x)$ or $\lim\limits_{x \to a^-} f(x)$ or $f(a-)$.

Note that $f$ is continuous at $a$ if, and only if, $\lim\limits_{x \downarrow a} f(x) = \lim\limits_{x \uparrow a} f(x) = f(a)$.

## 6. The Derivative

**Definition**: The function $f$ is said to be *differentiable at $x$* if the following limit exists:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In this case, the limit is called the derivative of $f$ at $x$, denoted by $f'(x)$ or $\dfrac{\mathrm{d}f}{\mathrm{d}x}$.

**Theorem**: If $f$ and $g$ are both differentiable at $x$, then:

1. for all constants $\alpha, \beta$, the function $\alpha f + \beta g$ is differentiable at $x$, and
$$(\alpha f + \beta g)' = \alpha f' + \beta g';$$
2. (**Product Rule**) the function $f \cdot g$ is differentiable at $x$ and
$$(f \cdot g)' = f' \cdot g + f \cdot g';$$
3. (**Quotient Rule**) if $g(x) \neq 0$, the function $\dfrac{f}{g}$ is differentiable at $x$, and
$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2};$$

4. if $y = f(x)$, and $f$ is invertible so that $x$ can also be written as a function of $y$, then
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}.$$

**Theorem** (**Chain Rule**): If $y$ is a function of $u$, where $u$ is a function of $x$, then
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

**Theorem** (**L'Hospital's Rule**): If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$
provided the second limit exists.