MAT1801 Mathematics for Engineers I

Sequences and Series

1 Sequences

Informally, a *sequence* is an infinite list of numbers, usually labelled as follows: $a_1, a_2, a_3, \ldots$. More precisely, a sequence is a function f from $\mathbb{N}$ to $\mathbb{R}$ such that $f(1) = a_1, f(2) = a_2, f(3) = a_3$ etc.

We shall denote a sequence by $\{a_n\}_{n=1}^{\infty}$ or simply by a_n.

The sequence is said to converge to a limit A if the terms in the sequence get closer and closer to A as n increases. More precisely, if the limit of the sequence is A, we can make the terms a_n as close to A as we wish by taking n sufficiently large.

Definition: The sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to a limit A if, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - A| < \varepsilon$ for all $n \geq N$.

If the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the limit A, we write $\lim\limits_{n\to\infty} a_n = A$.

If a sequence has a limit, then this limit is unique, and the sequence is said to be *convergent*. Otherwise, it is said to be *divergent*.

Note that adding or deleting a finite number of terms from a convergent sequence does not change the limit. Hence, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$ (where the limit exists).

Theorem: Let $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$. Then

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = A \pm B$;
2. $\lim\limits_{n\to\infty} (ca_n) = c\left(\lim\limits_{n\to\infty} a_n\right) = cA$;
3. $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right) = AB$;

4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$;

5. $\lim_{n\to\infty} a_n^c = \left(\lim_{n\to\infty} a_n\right)^c = A^c$, provided A^c exists;

6. $\lim_{n\to\infty} c^{a_n} = c^{\left(\lim_{n\to\infty} a_n\right)} = c^A$, provided c^A exists.

We write $\lim_{n\to\infty} a_n = \infty$ if the terms in the sequence grow arbitrarily large for larger n (more precisely, for all $M > 0$, there is a natural number N such that $a_n > M$ for all $n \geq N$).

Note that in this case the sequence is still a divergent sequence!

1.1 Bounded and Monotonic Sequences

Definition: If there exists a number M, independent of n, such that $a_n \leqslant M$ for all n, then the sequence is said to be *bounded above*, and M is said to be an *upper bound* for the sequence.

Definition: If there exists a number m, independent of n, such that $a_n \geqslant m$ for all n, then the sequence is said to be *bounded below*, and m is said to be a *lower bound* for the sequence.

Definition: If a sequence is both bounded above and bounded below it is said to be *bounded*.

Theorem: Every convergent sequence is bounded.

Note that not every bounded sequence converges e.g. $1, -1, 1, -1, 1, -1, \ldots$.

Definition: The sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *monotonic increasing* if $a_n \leqslant a_{n+1}$ for all n. If $a_n < a_{n+1}$ for all n, then it is said to be *strictly monotonic increasing.*

Definition: The sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *monotonic decreasing* if $a_n \geqslant a_{n+1}$ for all n. If $a_n > a_{n+1}$ for all n, then it is said to be *strictly monotonic decreasing.*

Theorem: 1. If a sequence is bounded above and monotonic increasing, then

it has a limit.
2. If a sequence is bounded below and monotonic decreasing, then it has a limit.

1.2 Some standard limits

1. $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$;
2. $\lim_{n\to\infty} \dfrac{n^p}{x^n} = 0$ for $|x| > 1$ and any p;
3. $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$;
4. $\lim_{n\to\infty} \left(1 + \dfrac{c}{n}\right)^n = e^c$;
5. $\lim_{n\to\infty} n^{-p} \ln n = 0$ for $p > 0$;
6. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

2 Series

Consider the sequence $\{a_n\}_{n=1}^{\infty}$, and define a new sequence as follows:

$$\begin{aligned} s_1 &= a_1; \\ s_2 &= a_1 + a_2; \\ s_3 &= a_1 + a_2 + a_3; \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n; \text{ etc.} \end{aligned}$$

The term s_n is called the nth partial sum of the series $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$.

Definition: If the sequence $\{s_n\}_{n=1}^{\infty}$ converges with limit S, then the series $\sum_{n=1}^{\infty} a_n$ is said to be *summable* or *convergent* and the limit $\lim_{n\to\infty} s_n = S$ is called the *sum* of the series. Otherwise, the series is said to be *divergent*.

Note: 1. If a series converges, then $\lim_{n\to\infty} a_n = 0$.

2. This is *not* a sufficient condition for convergence. For example, the series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

3. Multiplying every term in a series by the same nonzero constant does not affect whether the series converges or not.

4. Adding (or removing) a finite number of terms to (or from) a series does not affect whether the series converges or not.

2.1 Geometric Series

The sequence $a, ar, ar^2, ar^3, \ldots$ is called a geometric progression. The number r is called the *common ratio.*

The corresponding series $a + ar + ar^2 + ar^3 + \cdots$ is called a *geometric series.* The nth partial sum, s_n, of a geometric series is given by

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

when $r \neq 1$.

So when $|r| < 1$, the series converges to $\lim_{n\to\infty} s_n = \dfrac{a}{1 - r}$.

If $|r| \geqslant 1$, the series diverges.

2.2 Test for Convergence

1. **Comparison Test**

Suppose $0 \leqslant a_n \leqslant b_n$ for all $n = 1, 2, 3, \ldots$. Then

(i) if $\sum b_n$ converges, then $\sum a_n$ converges;

(ii) if $\sum a_n$ diverges, then $\sum b_n$ diverges.

2. **Ratio Test**

Suppose $a_n \geqslant 0$ for all n, and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$. Then

(i) if $r < 1$, the series converges;

(ii) if $r > 1$, the series diverges;

(iii) if $r = 1$, the test fails (i.e. no conclusion about the convergence of the series can be drawn).

3. **The nth Root Test**

Suppose $a_n \geqslant 0$ for all n, and $\lim_{n\to\infty} \sqrt[n]{a_n} = r$. Then

(i) if $r < 1$, the series converges;

(ii) if $r > 1$, the series diverges;

(iii) if $r = 1$, the test fails (i.e. no conclusion about the convergence of the series can be drawn).

4. **The Integral Test**

Suppose the function f is positive (i.e. $f(x) > 0$), continuous and monotonic decreasing[1] for $x \geqslant 1$, and $f(n) = a_n$ for all $n = 1, 2, 3, \ldots$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the limit $\lim_{N\to\infty} \int_1^N f(x)\,dx$ exists.

Example: By considering the function $f(x) = x^{-p}$ (for $p \neq 1$), find the values of p such that the series $\sum_{n=1}^{\infty} \frac{1}{x^p}$ converges.

5. **The Alternating Series Test**

An alternating series is a series whose terms are alternately positive and negative. Such a series converges if:

(i) $|a_{n+1}| \leqslant |a_n|$;

(ii) $\lim_{n\to\infty} a_n = 0$.

[1] A function f is said to be monotonic decreasing if $f(x) \geqslant f(y)$ whenever $x < y$.

2.3 Absolute and Conditional Convergence

Definition: The series $\sum a_n$ is said to be *absolutely convergent* if the series $\sum |a_n|$ converges.

Theorem: If the series $\sum a_n$ is absolutely convergent, then it is convergent (i.e. if $\sum |a_n|$ converges, then $\sum a_n$ converges).

If the series $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ is said to be *conditionally convergent.*