Chapter 1

Statements and Logic

1.0.1 Introduction

What is Mathematics? One answer is that mathematics is about solving problems, usually involving mathematical objects such as numbers, functions, equations, matrices, circles, lines and planes and so on.

Very often, finding the solution to a problem involves proving a statement (called a **proposition** or **theorem**) of the type 'Every polynomial is differentiable', 'Every nonsingular square matrix has a unique inverse', or 'There is no rational number whose square is 5'.

A mathematical proof is a convincing explanation that a certain statement is true (at least, it has got to convince other mathematicians!). We usually start with a statement that we know to be true, and then proceed to make logical deductions to find other true statements, culminating in the statement that we wish to prove.

It is important to note that every step in this chain has to be correct, otherwise we cannot claim that we have successfully proved the theorem.

Before we start to discuss different types of proofs, we have to look at statements and study how to work with them according to the rules of logic.

1.1 Statements

A sentence is called a **statement** if we can decide whether it is true or false.

Examples: (1) The sun is shining.
(2) A dog has four legs.
(3) For all real numbers x, $x^2 > 0$.

The following are not statements: Come here!
What time is it?
This statement is false.

Statements are usually denoted by the letters p, q, r etc. A statement about x, where x can take on several values, is sometimes denoted $p(x)$.

1.2 New statements from old: Negation

Given a statement p, we can form its **negation**, *not* p, denoted $\neg p$.

If p is the statement "The sun is shining" then $\neg p$ is the statement "The sun is not shining".

If p is false, then $\neg p$ must be true and vice-versa.

What is the negation of this statement: "$x > 5$"?

A **truth table** tells us when a statement is true and when it is false. The truth table for *not* is:

p	$\neg p$
T	F
F	T

1.3 New statements from old: Compound Statements

Another way of forming new statements is by joining two or more statements together. For example, "The sun is shining" and "Rain is falling" may be joined together to form the following compound statements:

The sun is shining *and* rain is falling.
The sun is shining *or* rain is falling.
If the sun is shining, *then* rain is falling.
The sun is shining *if and only if* rain is falling.

The truth value of these compound statements depends on the truth values of the simpler statements from which they are composed.

1.3.1 And

The statement "p and q" is denoted $p \wedge q$. It is true when **both** p and q are true, and is otherwise false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: "$2^2 = 4$ and 2 is a negative number".

1.3.2 Or

The statement "p or q" is denoted $p \vee q$. It is false when **both** p and q are false, and is otherwise true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: "$2^2 = 4$ or 2 is a negative number".

1.3.3 Implies

The statement "p implies q" or "if p, then q" is denoted $p \Rightarrow q$. It is false **only** when p is true and q is false, and is otherwise true. (Careful!)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that a true statement can only imply a true statement i.e. if $p \Rightarrow q$ and p is *true*, then q must be true. However, a false statement can imply both true and false statements i.e. if $p \Rightarrow q$ and p is *false*, then q could be either true or false.

Exercise: Let p be "That animal is a dog", and q be "That animal has four legs". Which of the following statements are true?

$p \Rightarrow q$	"If that animal is a dog, then it has four legs".
$q \Rightarrow p$	"If that animal has four legs, then it is a dog".
$\neg p \Rightarrow \neg q$	"If that animal is not a dog, then it does not have four legs".
$\neg q \Rightarrow \neg p$	"If that animal does not have four legs, then it is not a dog".

The statement $q \Rightarrow p$ is called the **converse** of the statement $p \Rightarrow q$.

The statement $\neg p \Rightarrow \neg q$ is called the **inverse** of the statement $p \Rightarrow q$.

The statement $\neg q \Rightarrow \neg p$ is called the **contrapositive** of the statement $p \Rightarrow q$, and it is always equivalent to $p \Rightarrow q$.

1.3.4 Equivalence

The statement "p if and only if q" or "p is equivalent to q" is denoted $p \Leftrightarrow q$. It is true when p and q have the same truth value, and is otherwise false.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example: "$2^2 = 4$ if and only if 2 is a negative number".

1.4 Quantifiers

Several mathematical statements contain phrases like "for all real numbers x" or "there exists an integer a". Phrases such as "for all", "for some" and "there exists" are called **quantifiers**.

The symbol $\forall$ denotes "for all", e.g. $\forall x \in \mathbb{R}, x^2 \geqslant 0$.

The symbol $\exists$ denotes "there exists" or "for some", e.g. $\exists x \in \mathbb{C}$ such that $x^2 = -5$ or $\exists x \in \mathbb{R},\ x^2 < x$.

There is a very simple rule which helps us to form the negation of a statement involving a quantifier. The negation of a statement of the type "$\forall x, p(x)$" is the statement "$\exists x, \neg p(x)$".

Example: The negation of "$\forall x \in \mathbb{R}, x^2 > 0$" is "$\exists x \in \mathbb{R}, x^2 \leqslant 0$".

Exercise: Which of the following statements is the negation of "All square matrices have an inverse"?

(i) No square matrices have an inverse.
(ii) A^{-1} does not exist for some square matrix A.
(iii) All square matrices do not have an inverse.
(iv) There exists at least one square matrix that does not have an inverse.

Important note: to prove that a statement of the type "$\forall x, p(x)$" is false, we need to find just *one* x for which $p(x)$ does not hold (i.e. a **counterexample**).

The negation of a statement of the type "$\exists x, p(x)$" is the statement "$\forall x, \neg p(x)$".

Example: The negation of "$\exists x \in \mathbb{R}, x^2 = -1$" is the statement "$\forall x \in \mathbb{R}, x^2 \neq -1$".

Exercise: What is the negation of the following statement: 'For all real numbers x, there exists an integer y such that $x < y$'?

Note that when combining several quantifiers in one statement, the order of the quantifiers may be important. The statements '$\forall x\ \exists y\ p(x,y)$' and '$\exists y\ \forall x\ p(x,y)$' are not equivalent.

1.5 Negating Compound Statements

There are two very important rules, known as **De Morgan's Laws**, that tell us how to find the negation of 'and' and 'or' statements:

(i) The negation of 'p and q' is 'not p **or** not q'.
(ii) The negation of 'p or q' is 'not p **and** not q'.

(Note that when we take the negation, 'and' becomes 'or' and vice versa.)

For example, the negation of 'f is one-to-one and onto' is 'f is not one-to-one or f is not onto'.

Exercise: What is the negation of the following statement: 'there exists x such that x is rational and $x^2 = 2$'?

What about the negation of 'if ..., then ...' statements? It is important to realize that '$p \not\Rightarrow q$' is **not** equivalent to any of the following:
(i) $\neg p \Rightarrow \neg q$; (ii) $\neg q \Rightarrow \neg p$; (iii) $\neg p \Rightarrow q$; (iv) $p \Rightarrow \neg q$; etc.

The rule is as follows: the negation of the statement $\forall x, p(x) \Rightarrow q(x)$ is $\exists x, p(x) \wedge \neg q(x)$. (If you wish, you may prove this by constructing a truth table for $p \wedge \neg q$.)

The value of x for which the implication is false is a counterexample to the statement $\forall x, p(x) \Rightarrow q(x)$.

For example, the negation of 'for all functions f, if f is continuous, then it is differentiable' is 'there exists a function f such that f is continuous and f is not differentiable'.

Exercise: What is the negation of the following statement: 'for all real numbers x and y, if xy is positive, then both x and y are positive'?

1.6 The truth value of a general statement

Truth tables help us to decide when a compound statement is true.

Example: When is the statement $(p \vee q) \wedge \neg p$ true?

Solution: Fill in the following truth table:

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$
T	T			
T	F			
F	T			
F	F			

1.7 Rules of Logic

Here is a list of some of the most common and useful rules of mathematical logic.

(i) $p \vee \neg p$ (Law of the Excluded Middle)

This means that either p or its negation must be true.

(ii) $\neg(p \wedge \neg p)$

This means that it cannot be the case that **both** p and its negation are true (in other words, either p or its negation is false).

(iii) $\neg\neg p \Leftrightarrow p$

The negation of the negation of p is equivalent to p.

(iv) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$ (Law of Contrapositive)

(v) $(p \wedge q) \Rightarrow p$

$p \Rightarrow (p \vee q)$

If both p and q and are true, then p is true (!).

If p is true then the statement 'p or q' is also true.

(vi) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ (Law of Syllogism)

(vii) $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ (De Morgan's Laws)

$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$

These have been explained in 1.5 above.

(viii) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ (Distributive Laws)

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

This rule tells us how to combine 'and' and 'or' in the same statement.

(ix) $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$

These rules, also known as **tautologies** (i.e. statements that are always true), may be verified using truth tables.

A **contradiction** is a statement that is always false. Clearly, any contradiction is the negation of a tautology, e.g. $p \wedge \neg p$.

1.8 Problems

1. Show that $(p \wedge q) \vee (\neg p \wedge \neg q) \Leftrightarrow (p \vee \neg q) \wedge (\neg p \vee q)$. (Hint: you may use a truth table or use the Distributive Laws.)

2. Let p/q be defined as $\neg p \wedge \neg q$.

   (a) Write down the truth table for p/q.

   (b) Show that $\neg p \Leftrightarrow p/p$ and that $(p \wedge q) \Leftrightarrow (p/p)/(q/q)$.

   (c) Find a way of expressing $p \vee q$ by using only the symbol / (but not any of $\neg, \vee, \wedge$ etc.).

3. Which of the following statements are true? Write down the negation of each one.

   (a) $\forall y \in \mathbb{R}$, $\exists x \in \mathbb{R}$ such that $x^2 = y$.

   (b) There exists an integer y such that, for all integers x, y is a factor of x.

   (c) For all rational numbers a and b, $a + b$ is also a rational number.

4. Are the following statements true? Find their negation, converse and contrapositive, and decide which are true :

   (a) For all real numbers x, y and z, if $x = y$ then $xz = yz$.

   (b) $\forall a, x, y \in \mathbb{R}, (x > y) \Rightarrow (ax > ay)$.

   (c) For all $a, b, c \in \mathbb{Z}$, if b is a factor of a and c is a factor of a, then bc is a factor of a.

5. Find the negation of the following statement about the function f and the fixed real number a:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R}, \text{ if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \epsilon.$$

   (This is the precise definition of the statement 'f is continuous at a'.)

6. Which (if any) of the following statements about $x \in \mathbb{Z}$ are true:
   (a) $\forall x, [x \text{ is even} \vee x \text{ is odd}]$ (b) $[\forall x, x \text{ is even}] \vee [\forall x, x \text{ is odd}]$
   (c) $\exists x, [x \text{ is even} \wedge x \text{ is odd}]$ (d) $[\exists x, x \text{ is even}] \wedge [\exists x, x \text{ is odd}]$?

7. There is a barber in Seville who shaves the beards of all those men that do not shave themselves. Who shaves the barber?