Introduction to Graph Theory — Sheet 1

Problems marked with an asterisk will be worked out in class.

Elementary

1. * Draw all twenty non-identical graphs on four vertices and three edges. How many of these are non-isomorphic? In general, how many non-identical graphs on n vertices and m edges are there? How many non-identical graphs on n vertices are there?

2. * Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote, respectively, the minimum and the maximum degree in the graph G. Show that

$$\delta \leq \frac{2m}{n} \leq \Delta.$$

3. * Show that the degrees in a graph cannot all be distinct. (Remember that a graph, unless otherwise stated, has no loops or multiple edges.)

4. An isomorphism $\phi : V(G) \to V(G)$ is said to be an *automorphism* of G.

   (a) Show that the set of automorphisms of G form a group under composition of functions. Denote this group by $\mathrm{Aut}(G)$.

   (b) Show that $|\mathrm{Aut}(G)|$ divides $n!$ and is equal to $n!$ iff $G \simeq K_n$ or $G \simeq \overline{K_n}$.

   (c) Find $\mathrm{Aut}(G)$ if G is a 6-cycle.

   (d) Consider the graph G in Figure 1. How many non-identical labellings of the vertices of G with the labels $\{1, 2, 3, 4, 5, 6\}$ are there? What is $|\mathrm{Aut}(G)|$? What is the relationship between 6! and these two results?

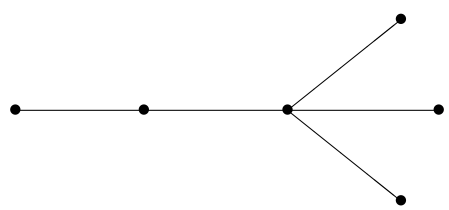

Figure 1: How many distinct labellings does this graph have?

(e) In general, what is the relationship between $|\mathrm{Aut}(G)|$, $n!$ (where n is the number of vertices of G), and the number of ways of labelling G with the labels $1, 2, \ldots, n$?

5. * Show that if a graph G is *self-complementary*, that is, $G \simeq \overline{G}$, then $n = 0 \bmod 4$ or $n = 1 \bmod 4$. Find a self-complementary graph on five vertices.

Medium

1. Remember that the distance between two vertices u, v is denoted by $d(u, v)$ and it is equal to the minimum length of a path joining u and v. Also, δ denotes the minimum degree.

   (a) Show that for any $u, v, w \in V$,

   $$d(u, w) \leq d(u, v) + d(v, w).$$

   (b) * Show that any two longest paths in a graph must have a common vertex.

   (c) * Show that if G is simple then it must have a path of length k for every $k \leq \delta$

   (d) * Show that if G is simple and $\delta > \lfloor n/2 \rfloor - 1$, then G is connected. Find a disconnected $\frac{n}{2} - 1$)-regular graph for even n.

2. * Show that if G is simple and bipartite then

   $$m \leq \frac{n^2}{4}.$$

3. Show that in a party of six or more people either there are three persons who know each other or there are three person who are mutual strangers. (Assume that if x knows y then y knows x.)

4. * Prove that if G is simple and $\delta \geq 2$ then it contains a cycle of length $\geq \delta{+}1$. [Hint: Take a longest path and consider the degree of an endvertex of this path.]

5. Show that if G is simple and connected but not complete then it contains three vertices u, v, w such that $uv, vw \in E(G)$ but $uw \notin E(G)$.

6. Let $c(G)$ denote the number of components of G.

   (a) Show that

   $$c(G) \leq c(G - e) \leq c(G) + 1$$

   for every edge e in $E(G)$.

   (b) Suggest a similar inequality for $c(G - v)$ where v is a vertex in $V(G)$.

(c) * Show that if each degree in G is even and G is disconnected, then there exists no edge e in $E(G)$ such that $G - e$ is disconnected.

(d) Show that if G is connected and each degree is even, then

$$c(G - v) \leq \frac{1}{2} \deg(v),$$

for every vertex $v \in V(G)$.

Harder

1. The *girth* $\gamma = \gamma(G)$ of G is the length of a shortest cycle in G. If there are no cycles we let $\gamma = \infty$. Prove that

   (a) If G is r-regular and $\gamma = 4$ then $n \geq 2r$ and there is exactly one such graph (up to isomorphism) on $2r$ vertices.

   (b) If G is r-regular and $\gamma = 5$ then $n \geq r^2 + 1$. Find such a graph for $r = 2, 3$. [Note: It is known that such graphs can only exist if $r = 2, 3, 7$ and possibly 57.]

2. Let G be simple nd let p be an integer such that $1 < p < n - 1$. Show that if $n \geq 4$ and all induced subgraphs of G on p vertices have the same number of edges, then either $G \simeq K_n$ or $G \simeq \overline{K_n}$.

3. Let A and B be, respectively, the adjacency matrix and the incidence matrix of a graph G. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A.

   (a) What is the value of every column sum of B? And of A

   (b) Show that the number of $[v_i, v_j]$-walks of length k in G is given by the i, j-entry of A^k.

   (c) Show that if G is simple, then the entries on the diagonals of both BB^t and A^2 are the degrees of the vertices of G.

   (d) Why is each eigenvalue of A real?

   (e) Show that

      i. $\sum \lambda_i = 0$.
      ii. $\sum \lambda_i^2 = 2m$, where m is the number of edges of G.
      iii. $\sum \lambda_i^3 = 6t$, where t is the number of triangles of G.
      iv. For each λ_i, $|\lambda_i| \leq \sqrt{2m(n-1)/n}$.