Introduction to Graph Theory — Sheet 3

1. The *line graph* $L(G)$ of a graph G is a graph whose vertices are the edges of G and such that two vertices are adjacent in $L(G)$ iff the corresponding edges are e=adjacent in G. Show that

   (a) Show that if G is Eulerian then its line graph $L(G)$ is Hamiltonian. *Give a counterexample to show that the converse is false.

   (b) Show that if G is Hamiltonian then its line graph $L(G)$ is Hamiltonian. *Give a counterexample to show that the converse is false.

2. This describes an application of Eulerian graphs. Do you recognise which real-life situation this refers to?

   Suppose that W is a word composed from the alphabet $\{A, C, G, T\}$. Suppose that W contains fourteen letters. Yous are not given W but you are given instead all its subwords of length 3. These subwords, given in an arbitrary order which need not be the order they appear in W are:

   $$AAA, AAC, ACA, ACG, ACT, CAA$$

   $$CAC, CGC, CTT, GCA, TAA, TTT.$$

   Can you determine W from these fragments? Is W the only word with these fragments?

   Solve the problem by drawing an appropriate digraph D and finding an Eulerian directed trail in D. (You can find this problem and its solution in my Networks course notes.)

3. *Prove that if G is a Hamiltonian graph then, for every set of vertices S of G the number of components $c(G - S)$of $G - S$ is at most $|S|$. Show by an example that this necessary result is not sufficient, that is, a graph could have the property that $c(G - S) \leq |S|$ for all sets of vertices but still fail to be Hamiltonian.

4. Two questions about Ore's Theorem

   (a) *Give an example to show that Ore's Theorem is best possible in the sense that there could be a graph such that $\deg(u) + \deg(v) \geq n - 1$ for all non-adjacent vertices u, v which still fails to be Hamiltonian.

   (b) *Show by an example that the sufficient condition of Ore's Theorem is not necessary, that is, there could be a a Hamiltonian graph for which it is not true that $\deg(u) + \deg(v) \geq n$ for all non-adjacent vertices u, v.

5. *Prove that a Hamiltonian graph is 2-connected, but the converse does not hold.