

Introduction to Graph Theory — Sheet 4 v2

Most of these problems are elementary, meaning that they do not require any deep knowledge of graph theory or of any other area of mathematics. A few, however, involve some ingenious little tricks which might be new to you. For these we have given hints, sometimes more than one. The best way to attempt these questions is first to give yourself some time to work out the problem yourself – you might even find a better solution than that suggested by the hint. After this, look at the hint(s) (even if you think you have solved the problem, compare your method with the hint). Having spent some time trying hard to solve the problem yourself, seeing the hint will teach you much more than just going straight to it.

If you lack time, try doing first those which have two asterisks. We shall try, if we have time, to cover in class all those questions with at least one asterisk.

1. **Find orderings of the graph of the cube for which the greedy algorithm requires 2, 3, and 4 colours respectively.
2. **The graph M_r , $r \geq 2$, is obtained from the cycle graph C_{2r} by adding extra edges joining each pair of opposite vertices. Show that
 - (a) M_r is bipartite (hence $\chi(M_r) = 2$) when r is odd;
 - (b) $\chi(M_r) = 3$ when r is even and $r \neq 2$;
 - (c) $\chi(M_2) = 4$.
3. ** Using the full Brooks's Theorem, show that the Petersen graph has chromatic number 3.
4. **Show that if G is an r -regular graph with n vertices then

$$\chi(G) \geq \frac{n}{n-r}.$$

[Hint: Let C_1, \dots, C_χ be the colour classes of a χ -colouring of G . Then $n = \sum |C_i|$. Can you see that $|C_i| \leq n - r$?

5. *(More difficult.) Show that if G is a simple graph on n vertices and m edges then

$$\chi(G) \geq \frac{n^2}{(n^2 - 2m)}.$$

[Hint: Let $C(v_i)$ denote the colour class containing vertex v_i . As in the hint to Problem 4, $|C(v_i)| \leq n - \deg(v_i)$. Also, $\sum_1^n (n - \deg(v_i)) = n^2 - 2m$, therefore $\sum_1^n |C(v_i)| \leq n^2 - 2m$. If the sizes of the χ distinct classes are s_1, \dots, s_χ , this gives $\sum_1^\chi s_i^2 \leq n^2 - 2m$. Use the inequality $(\sum_1^\chi s_i)^2 \leq \chi \sum_1^\chi s_i^2$. This last inequality can be obtained using the Cauchy-Schwarz Inequality which you might have encountered in an analysis course.]

6. *Show that if G has the property that any two odd cycles have at least a vertex in common then $\chi(G) \leq 5$.

[Hint: Suppose for contradiction that the chromatic number of G is more than 5. Let C_1, \dots, C_6 be six distinct colour classes. Then there must be an odd cycle amongst the vertices of C_1, C_2, C_3 (why?). Similarly, there must be an odd cycle amongst the vertices of C_4, C_5, C_6 . But this gives a contradiction.]

[Alternative hint: Let C be a shortest odd cycle in G ; C cannot have any chord (why?). So we can colour the vertices of C with three colours. Removing the vertices of C from G leaves us with a graph which we can be coloured with another 2 colours (why?). Combine the two colourings.]

7. (More difficult) Let $\chi = \chi(G)$ and $\bar{\chi} = \chi(\bar{G})$.

- (a) **Prove that $\chi\bar{\chi} \geq n$;

[Hint: Suppose that a vertex v has colour i in a χ -colouring of G and colour j in a $\bar{\chi}$ -colouring of \bar{G} . Let v be given the colour (i, j) . Show that this new colouring is a proper colouring of K_n .]

[Alternative hint: As in Question 4, let C_1, \dots, C_χ be the colour classes of a χ -colouring of G . A moment's thought should convince you that $|C_i| \leq \bar{\chi}$. Then proceed as in Question 4.]

- (b) *Deduce that $\chi + \bar{\chi} \geq 2\sqrt{n}$;
 (c) Use induction to show that $\chi + \bar{\chi} \leq n + 1$;
 (d) Deduce that $\chi\bar{\chi} \leq \frac{1}{4}(n^2 + 2n + 1)$.

*Give examples to show that all these bounds can be attained.