Introduction to Graph Theory — Sheet 5

Try to do at least all those marked with an asterisk. Most of them are very easy.

1. *Show that the graph shown in the figure is planar.

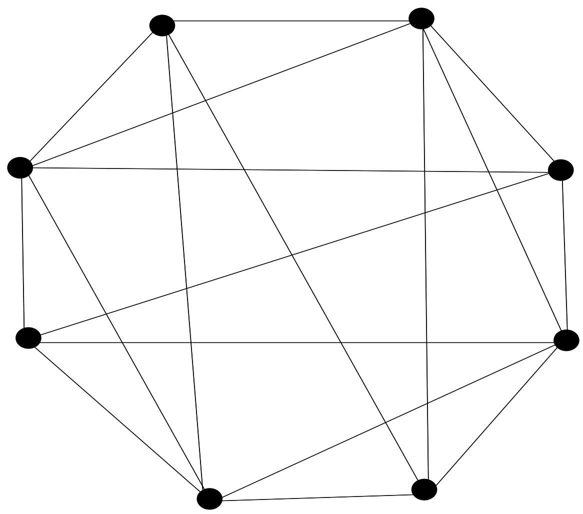

Figure 1: Is this graph planar?

2. *Find a non-planar graph which is *not* a subdivision of K_5 or $K_{3,3}$. Why does this not contradict Kuratowski's Theorem?

3. *Which complete graphs and complete bipartite graphs are planar?

4. Look up "stereographic projection" in any of the suggested texts (or via Google). The important point for us here is that any plane graph can be translated into an embedding of the sphere and vice-versa. Embedding a planer graph in the plane is therefore considered to be equivalent to embedding it on the sphere. The unbounded face of a plane graph corresponds to the face which includes the North Pole on the sphere. Therefore, by taking a plane graph, embedding it on the sphere by stereographic projection. Rotating the sphere so that the North Pole is inside another face, and embedding back onto the plane by stereographic projection, one can re-draw the original graph such that any other chosen face is the unbounded face.

5. *Let G be a cubic (all degrees equal to 3) plane graph and let f be the number of faces of G and m the number of edges. Show that

$$f = \frac{m}{3} + 2.$$

Now suppose that G has only two types of faces: those bounded by a cycle of length 5 and those bounded by a cycle of length 6. Let f_5 denote the number of pentagonal faces and f_6 the number of hexagonal faces. Using the facts that $f_5 + f_6 = f$ and $5f_5 + 6f_6 = 2m$ (why?) show that $f_5 = 12$. Verify your result by looking at the stitching of a football and counting the number of pentagons! Here the number of vertices is 60.

These graphs are not not just an idle pastime. In 1985 a new allotrope of carbon was found: fullerenes. They are molecules composed entirely of carbon, in the form of a hollow sphere, ellipsoid, or tube. Their molecular structure is described by the planar graph G. The most common fullerene is C_{60} whose molecular structure looks like the stitching of a football. Sometimes you can see a model of the dual of C_{60} erected in the roundabout just outside the University. Look up some information on fullerenes.

6. *Let G be a planar graph on n vertices, m edges and with girth (size of smallest cycle) g. Suppose that G has no cut-edges. Show that

$$m \leq \frac{g}{g-2}(n-2).$$

   Deduce that the Petersen graph is non-planar.

   Find, inside the Petersen graph, a subdivision of K_5 or $K_{3,3}$. Note that the Petersen graph is not, itself, a subdivision of K_5 or $K_{3,3}$.

7. Show that if G is planar and $n \geq 11$ then $\overline{G}$ is non-planar.

   Find a planar graph G with $n = 8$ such that $\overline{G}$ is also planar.

8. *Consider the following simple proof that the chromatic number of a planar graph G is ≤ 6:

   > Induction on n, the number of vertices. Let v be a vertex with minimum degree, hence at most 5. By the induction hypothesis, colour $G - v$ using the six colours $\{1, 2, \ldots, 6\}$. One of these six colours, say j, is not used on any of the neighbours of v since there are at most five neighbours. Put back the vertex v and colour it j. We therefore have a 6-colouring of G.

   Explain why this argument cannot be used to prove that the chromatic number of any graph G is at most $\delta + 1$, where δ is the minimum degree in G.

9. Let $S = \{x_1, \ldots, x_n\} \subset \mathbb{R}$, $n \geq 3$, such that the distance between any two points of S is at least 1. Show that there are at most $3n - 6$ points of S at distance exactly 1.