MAT 2402: Networks — Solutions to selected exercises

Sheet 1

2: Handshaking Lemma :-

$$\sum \deg(v) = 2m$$

$$\therefore \quad \delta \cdot n \le \sum \deg(v) = 2m$$

and

$$2m = \sum \deg(v) \le \Delta n$$

$$\therefore \quad \delta \le \frac{2m}{n} \le \Delta$$

3: H'shaking Lemma :- $2m = \sum \deg(v) = r \cdot n$

$$\therefore \quad m = \frac{rn}{2}.$$

Complete graph K_n :- All degrees are equal to $r = n-1$

$\therefore m = \frac{n(n-1)}{2}$ by above.

__Alternatively__ No. of edges of K_n

$= $ no. of ways of choosing 2 elements (vertices) out of n

$= \binom{n}{2} = \frac{n(n-1)}{2}$.

__Note__: This number is the maximum number which a graph on n vertices can have (provided no multiple edges or loops are allowed).

6: Let the number of components be c.

Suppose the components are $C_1, C_2, \dots, C_c$ and that C_i spans n_i vertices (clearly $\sum_1^c n_i = n$).

Then C_i has $n_i - 1$ edges, since each component is a tree.

$$\therefore \quad k = \sum_1^c (n_i - 1) = \sum_1^c n_i - \sum_1^c 1 = n - c$$

$$\therefore \quad c = n - k, \text{ as req'd.}$$

Sheet 0

The ranking based on points is

Player 2 — 3 points
Players 1, 3, 4 — 2 points
Player 5 — 1 point.

The column stochastic matrix corresponding to A is

$$S = \begin{bmatrix} 0 & 0 & 1/2 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/3 \\ 1/2 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Verify that $\begin{pmatrix} 2/3 \\ 1 \\ 2/3 \\ 2/3 \\ 1 \end{pmatrix}$ is an eigenvector of S corresponding to the eigenvalue 1.

Therefore the eigenvector does not break the tie between players 1, 3 and 4, but it places player 5 in joint top position.

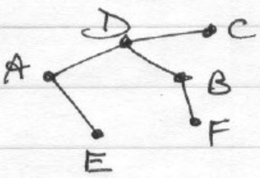
This is because player 5's only point was obtained against player 2 who had 3 points. The importance of this point (inherited from the importance of player 2) makes it more valuable than the two points of players 1, 3 or 4.

Sheet 2

1.

Latest	A	B	C	D	E	F	Next	New edge
	0	∞	∞	∞	∞	∞		
A		5	8	3	4	9	D	AD
D		4	6		4	9	B	BD
B			6		4	8	E	AE
E			6			8	C	CD
C						8	F	BF

Therefore a shortest route from A to F is ADBF of length $3+1+4=8$
The following spanning tree (obtained from last column) contains ~~all~~ shortest routes from A to every other vertex.



2. Kruskal's Alg gives:-

de, xc, bc, xb, xa, fa

with weight 20

4.

latest	1	2	3	4	next	new edge
	0	∞	∞	∞	1	
1		2	1	∞	3	13
3		2		2	2	12
2				2	4	34
4						

$\therefore$ Dijkstra's Alg. gives shortest distance 2 when, in fact, shortest route is 1-2-3-4 with length $2+(-2)+1=1$. Dijkstra is "too greedy" when there are −ve edges.

6 Graphs Minimise $\sum_j w_j x_j$

Subject to:-

$$\sum_j b_{1j} x_j = \sum_j b_{nj} x_j = 1$$

$$\sum_j b_{ij} x_j \le 2, \qquad i = 2, 3, \dots, n-1$$

$$x_j = 0 \text{ or } 1$$

Digraphs

Minimise $\sum_j w_j x_j$

Subject to

$$\sum_j b_{1j} x_j = 1$$

$$\sum_j b_{nj} x_j = -1$$

$$\sum b_{ij} x_j = 0 \qquad i = 2, 3, \dots, n-1$$

$$x_j = 0 \text{ or } 1.$$

7 Digraph with vertices $v_1, v_2, \dots, v_6$. Vertex v_i denotes beginning of year i.
There is an arc $v_i v_j$ whenever $i < j$, with weight c_{ij}.

c_{ij} = cost of owning & operating car from beg of year i to beg of year j if new car is bought at beg. of year i and traded in at beg of year j
= mainten cents for yrs $i, i+1, \dots, j-1$
+ cost of new car at beg of year i
− trade in for old car at beg of yr j.

~~(cont)~~ (7 cont.)

$\therefore\ c_{12} = 2+12-7 = 7$

$c_{13} = 2+4+12-6 = 12$

$c_{14} = 2+4+5+12-2 = 21$

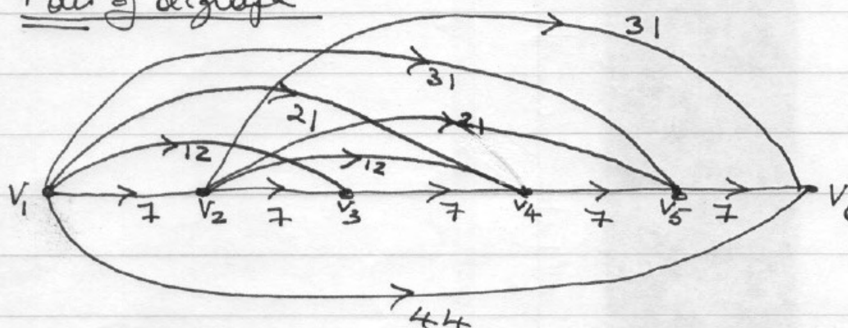
$c_{15} = \quad \ldots \quad = 31$

$c_{16} = \quad \ldots \quad = 44$

$\vdots$

$c_{25} = 2+4+12-6 = 12$

Part of digraph

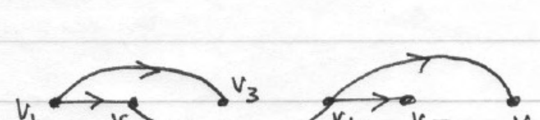


The whole distance matrix: (Easier to construct & work with than a picture of the digraph)

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 7 & 12 & 21 & 31 & 44 \\ 2 & & 0 & 7 & 12 & 21 & 31 \\ 3 & & & 0 & 7 & 12 & 21 \\ 4 & & & & 0 & 7 & 12 \\ 5 & & & & & 0 & 7 \\ 6 & & & & & & 0 \end{array}$$

Dijkstra's Algorithm

latest	V_1	V_2	V_3	V_4	V_5	V_6	next	new arc
	<u>0</u>	∞	∞	∞	∞	∞	V_1	
V_1		<u>7</u>	12	21	31	44	V_2	V_1V_2
V_2			<u>12</u>	19	28	38	V_3	V_1V_3
V_3				<u>19</u>	24	33	V_4	V_2V_4
V_4					<u>24</u>	31	V_5	V_4V_5
V_5						<u>31</u>	V_6	V_4V_6
V_6								



Optimal policy given by path $V_1 \to V_2 \to V_4 \to V_6$, i.e. trade in at beginning of yr 2, yr 4 and yr 6.

8: Vertices 0, 4, 8, 12

Arc ij for every $i < j$

Cost of ij = min. cost of storing all books of height $> i$ & $\leq j$ in one single shelf.

$\therefore C_{04}$ = cost of storing all books of height 4 on same shelf

$= 200 \times 4 \times \frac{1}{2} \times 50 + 23\,000 = 43\,000$

$C_{08} = (200+100) \times 8 \times \frac{1}{2} \times 50 + 23000 = 83000$

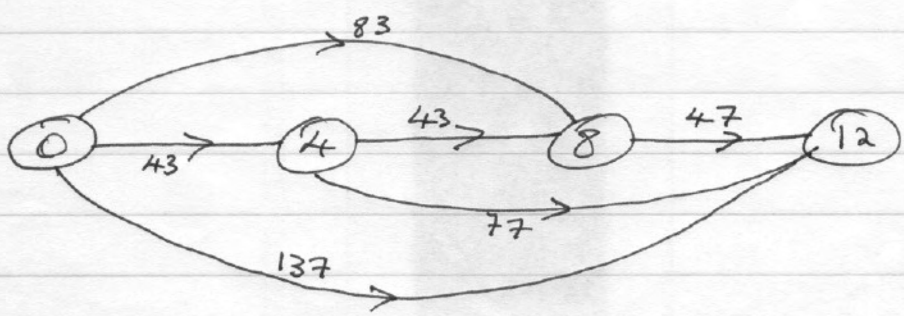
$C_{012} = (200+100+80) \times 12 \times \frac{1}{2} \times 50 + 23000 = 137\,000$

$C_{48} = 100 \times 8 \times \frac{1}{2} \times 50 + 23000 = 43\,000$

$C_{412} = (100+80) \times 12 \times \frac{1}{2} \times 50 + 23000 = 77000$

$C_{812} = 80 \times 12 \times \frac{1}{2} \times 50 + 23\,000 = 47\,000$

Digraph



Dijkstra gives

$\therefore$ shortest path is $0 \to 4 \to 12$ $\therefore$ store all 4" books on same shelf (4" high) and all 8" & 12" books on same shelf (12" high).

9: (b) Let d_{ij} be the length of a shortest path from i to j and p_j the population of j.

$\therefore$ if hospital is built in town i, total distance travelled

$$= \sum_j d_{ij} p_j \quad (\text{note that } d_{ii} = 0).$$

Build hosp in town :–	1	2	3	4	5	6	7
$\sum_j d_{ij} p_j$:–	1364	1017	1050	920	984	1279	1200

$\therefore$ build hospital in town 4.

<u>Note</u> If A is the full distance matrix, then the above row is the column of the multiplication $A\underline{p}$ where $\underline{p} = (p_1, p_2, \ldots, p_7)^t$.

9c

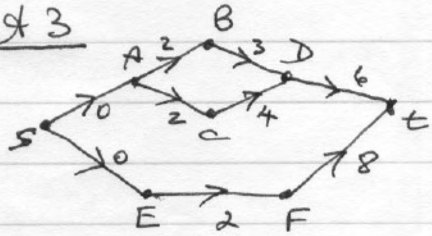
	Possible pairs of locations for schools									
i	1	1	1	1	3	3	3	4	4	6
j	3	4	6	7	4	6	7	6	7	7
$\max_k \left\{ \min \{ d_{ik}, d_{jk} \} \right\}$	7	8	6	7	5	8	8	4	5	8

↑ Choose this option.

(Best way to obtain the entries in the last row is to use the matrix A, fix rows i, j, then for every other column find the minimum value of its intersection with the two rows — then take the maximum over all the columns.)

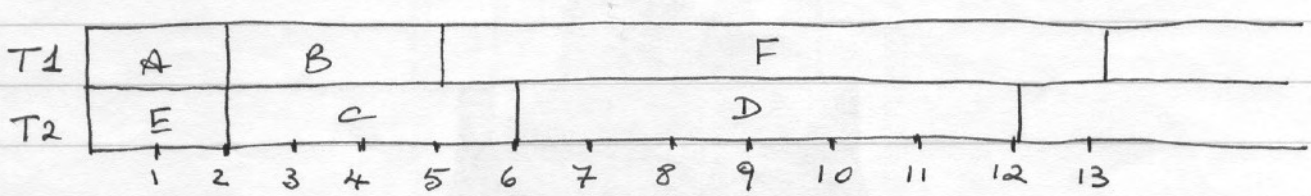
Sheet 3

2.



Activity	s	A	B	C	D	E	F	t
Earliest start time	0	0	2	2	6	0	2	12
Latest start time	0	0	3	2	6	2	4	12

Critical path scheduling for two teams



This gives an overall duration for the project of 13.

Now, total durations are $2+3+\cdots+8 = 25$.

$\therefore$ optimal time cannot be less than $25/2 = 12\frac{1}{2}$ days ($\because$ 2 teams)

But no two teams can work on same project,

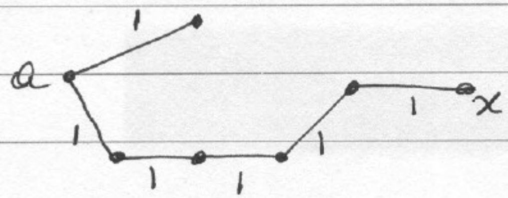
$\therefore$ optimal time has to be a whole no. of days

$\therefore$ opt. time $\geq \lceil 12\frac{1}{2} \rceil = 13$.

$\therefore$ above schedule is optimal for two teams.

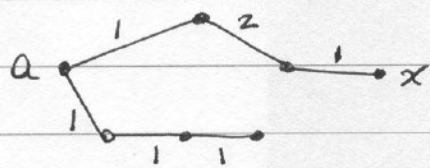
Sheet 2 No 3

Kruskal's Algorithm gives the following tree:



This is a minimum weight tree (weight = 6) but the distance it gives (distance = 5) between a and x is not minimum.

Dijkstra's Algorithm gives the following tree:



This gives the minimum distance between a and x (distance = 4) but it is not a minimum weight tree (weight = 7).

4. Applying the formulae gives the following mean durations and variances:

Act	A	B	C	D	E	F
Mean dur	6	9	8	7	10	12
Variance	1.78	1.78	2.78	4	0.44	1

Critical path turns out to be

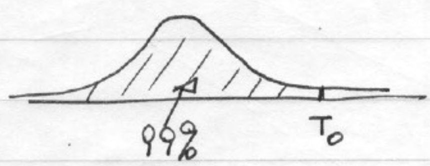
$$cp = sBDEFt$$

If T is ~~[illegible]~~ duration of cp, then

$$E(T) = 9+7+10+12 = 38 = \mu$$

$$Var(T) = \text{- - - -} = 7.22 \quad \therefore \text{st. dev} = \sqrt{7.22} = 2.69 = \sigma$$

Assume T is normally distributed. We require T_0 such that $Prob(T \leq T_0) = 0.99$



From tables this value of T_0 is 2.33.

$\therefore$ standardising T_0 gives:-

$$\frac{T_0 - \mu}{\sigma} = 2.33$$

$$\therefore T_0 = 44.3$$

$\therefore$ work should start 45 days before 1st December.

5. Let x_i denote the earliest starting time of activity i

& y_i - - - - amount by which activity is is to be crashed.

$\therefore$ minimise $30y_A + 15y_B + 20y_C + 40y_D + 20y_E + 30y_F + 40y_G$

subject to:-

$0 \leq y_A \leq 2$

$0 \leq y_B \leq 3$

$\vdots$

$0 \leq y_G \leq 1$

$x_B \geq x_A + 5 - y_A$

$x_C \geq x_B + 8 - y_B$

$x_D \geq x_B + 8 - y_B$

$x_E \geq x_B + 8 - y_B$

$x_F \geq x_E + 4 - y_E$

$x_G \geq x_C + 10 - y_C$

$x_G \geq x_F + 6 - y_F$

$x_t \geq x_G + 3 - y_G$

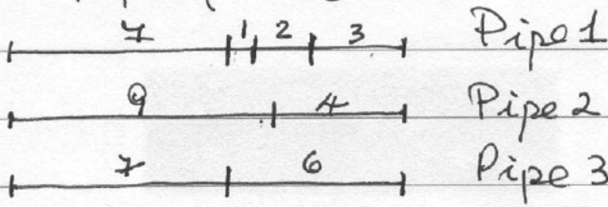
$x_t \geq x_D + 5 - y_D$

And: $x_t - x_A \leq 20$

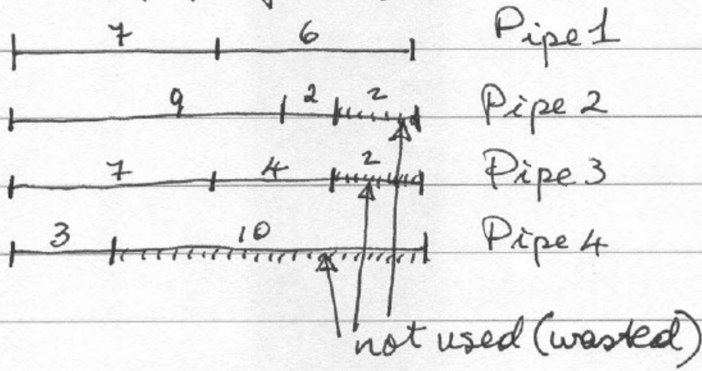
S7A

Sheet 3

3(b) With pipe of length 1



Without pipe of length 1

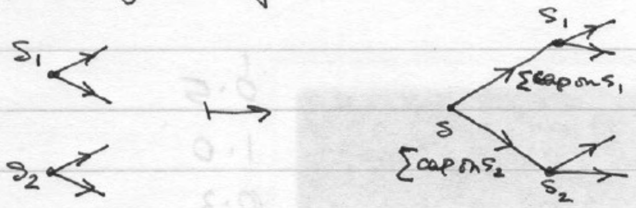


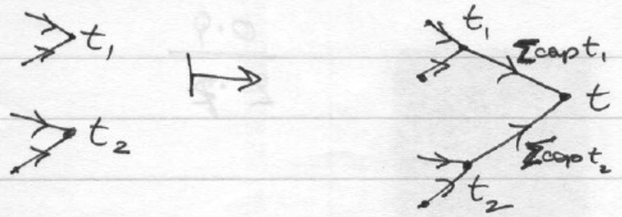
6- Optimal schedule : ~~⟨1, 2, 3, 4, ...~~

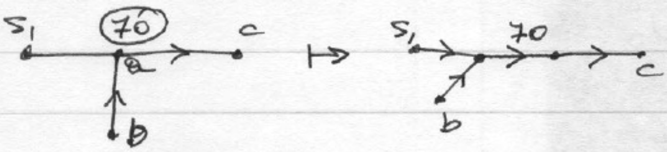
$\langle 2, 4, 1, 3, 7, 5, 6 \rangle$

Sheet 4

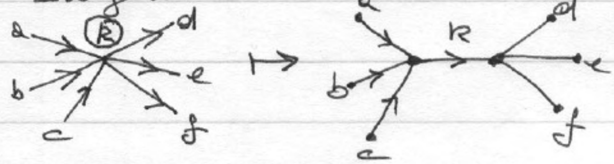
1. Make the following transformations:-

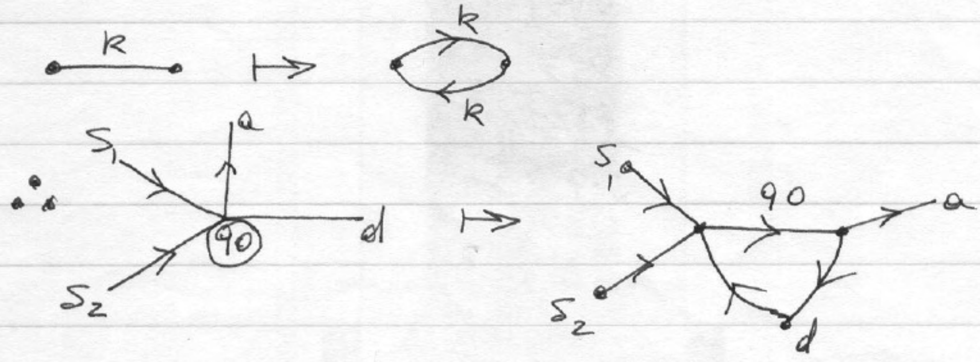
(i) 

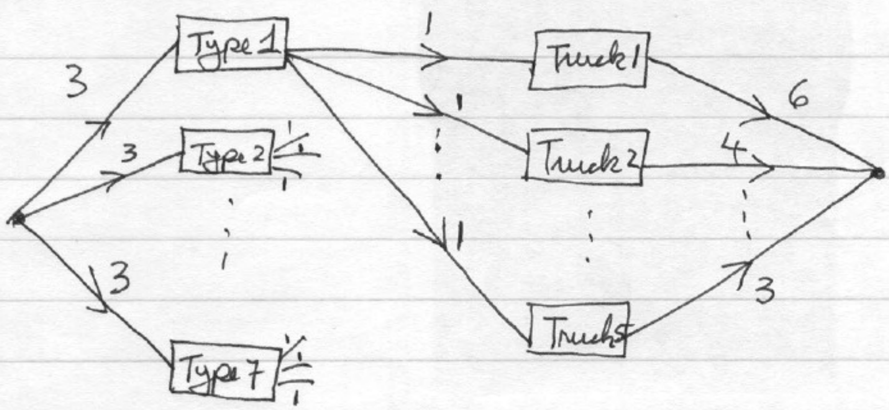
(ii) 

(iii) 

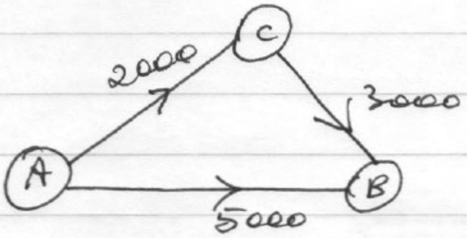
In general:-



(iv) 

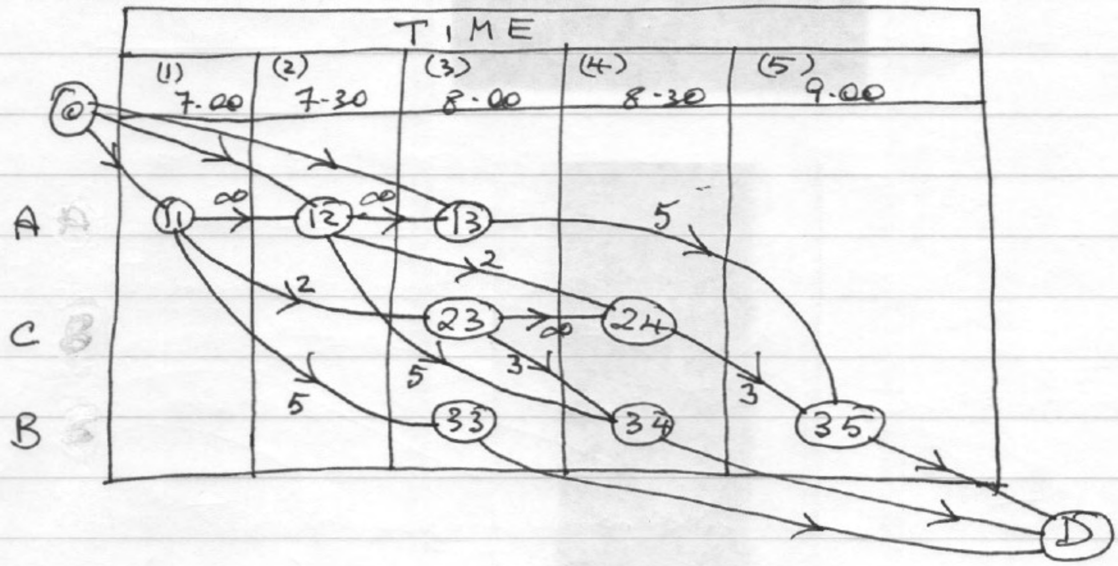
2. 

(3)

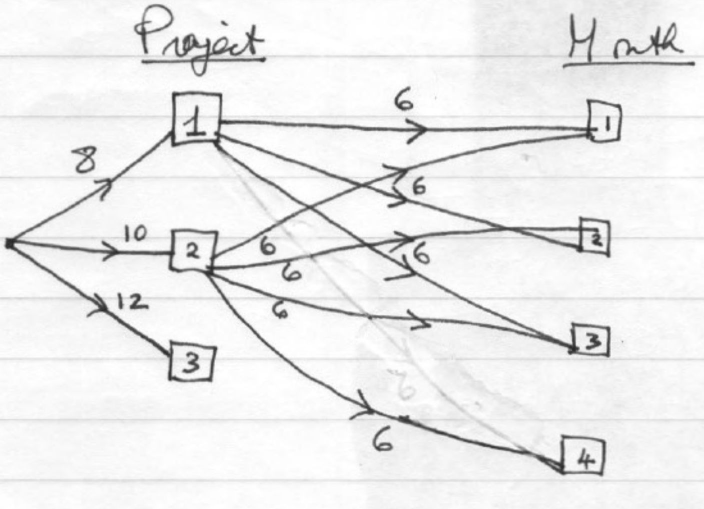


Time/space network

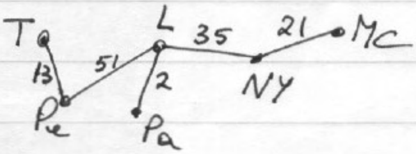
city



(4)



Sheet 5 (1) The following is a MWST



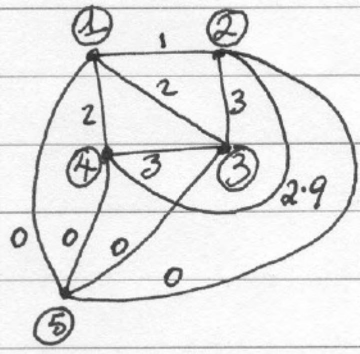
DFS starting from L :- L, Pe, T, Pe, L, Pa, L, NY, MC, L.

Omitting repetitions :- L, Pe, T, Pa, NY, MC, L

Total length = 238,000 miles.

Distances obey Δ ineq. $\therefore$ if ℓ_0 is optimal, $238{,}000 \le \ell_0 \therefore \ell_0 \geq 11{,}9$

2:



Solve TSP and then remove vertex 5 from the solution.

3: Vertices 0, 1, 2, 3, 4; vertex i represents sheet i, with vertex 0 representing the initial/end position.

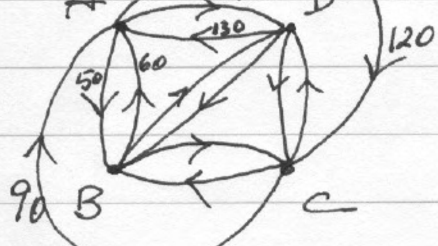
Length of arc ij is length of material wasted if sheet j is immediately cut after sheet i.

i \ j	0	1	2	3	4
0	—	0.3	0.4	0.2	0.7
1	0.3	—	0.7	0.5	0
2	0.2	0.5	—	0.4	0.9
3	0.5	0.8	0.9	—	0.2
4	0.1	0.4	0.5	0.3	—

These are the weights on the ~~edges~~ arcs of the digraph.

Find a minimum weight spanning directed cycle (TSP for digraphs). This gives an optimal schedule for cutting the patterns.

4: Directed graph with vertices A, B, C, D such that length of arc XY equals duration of Y preceded by X:—



(not all arc weights are shown)

Solve the TSP problem for the digraph to get an optimal schedule.