UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc.,B.Ed.,Year I / B.Sc.(I.T.) Year II
June Session 2003

MAT1401 Discrete Methods **21 June 2003**

0900–1130

Answer FOUR questions.

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1. (a) Let $A_1, A_2, \ldots, A_n$ be finite sets, and let α_i denote the sum of the cardinalities of the intersections of the sets taken i at a time. Write down the Inclusion-Exclusion Formula giving

$$|A_1 \cup A_2 \cup \ldots \cup A_n|$$

as a sum involving the α_i.

(b) Suppose there are m boxes labelled $1, 2, \ldots, m$ respectively and n balls $(m \geq n)$ labelled $1, 2, \ldots, n$ respectively. The balls are put into the boxes such that no two balls are in the same box and no ball labelled i is put into box i, for all $1 \leq i \leq n$. Show that the number of ways in which this can be done is

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(m-i)!}{(m-n)!}.$$

2. A set of 14 identical golf balls is partitioned into five parts: two parts have 4 balls each while each of the other parts have 2 balls. The balls are to be coloured using of the two colours red or blue, but balls in the same part of the partition are to be coloured the same. Obtain the generating function for the number of ways in which the colourings can be carried out.

What is the total number of possible colourings?

What is the total number of possible colourings if six balls are to be coloured red and eight balls are to be coloured blue?

3. Let q_n be the number of words of length n in the alphabet $\{a, b, c, d\}$ which contain an odd number of b's. Show that

$$q_n = 2q_{n-1} + 4^{n-1}, \qquad (n \geq 2)$$

and hence find q_n.

4. Solve the following recurrence relations, given that, for each, $a_0 = a_1 = 0$.

(i) $a_n - 5a_{n-1} + 6a_{n-2} = 5 \qquad (n \geq 2)$;

(i) $a_n - 3a_{n-1} + 2a_{n-2} = 5 \qquad (n \geq 2)$.

5. (a) Let $p(n)$ denote the number of partitions of the positive integer n and let $p(n|\mathcal{P})$ denote the number of partitions of n having property $\mathcal{P}$. Write down the generating functions of each of the following

(i) $p(n)$;

(ii) $p(n|$ all parts are distinct$)$;

(iii) $p(n|$ all parts are odd$)$;

(iv) $p(n|$ no part appears more than twice$)$;

(v) $p(n|$ no part is a multiple of 3$)$.

Show that

$$p(n|\text{ all parts are distinct}) = p(n|\text{ all parts are odd})$$

and

$$p(n|\text{ no part appears more than twice}) = p(n|\text{ no part is a multiple of 3}).$$

[You may need to use $1+y = (1-y^2)/(1-y)$ and $1+y+y^2 = (1-y^3)/(1-y)$.]

(b) Using Ferrers diagrams, show that

(i) The number of partitions of $3n$ into n parts is equal to the number of partitions of $2n$ into at most n parts.

(ii) The number of partitions of n into exactly m parts is equal to the number of partitions of n such that the largest part equals m.