UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc.,B.Ed.,Year I / B.Sc.(I.T.) Year II
June Session 2004

MAT1401 Discrete Methods **14 June 2004**

Time allowed: 2 hours 30 minutes — 0900–1130

Answer FOUR questions.

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1. A set of 14 identical golf balls is partitioned into five parts: two parts have 4 balls each while each of the other parts have 2 balls. The balls are to be coloured using the two colours, red or blue, but balls in the same part of the partition are to be coloured the same. Obtain the generating function for the number of ways in which the colourings can be carried out.
[This generating function should be an expression in the two variables r and b such that the coefficient of $r^i b^j$ equals the number of ways in which the balls can be coloured such that i balls are red and j balls are blue.]

What is the total number of possible colourings?

What is the total number of possible colourings if six balls are to be coloured red and eight balls are to be coloured blue?

2. Let q_n be the number of words of length n in the alphabet $\{a, b, c, d\}$ which contain an odd number of b's. Show that

$$q_n = 2q_{n-1} + 4^{n-1}, \qquad (n \geq 2)$$

and hence find q_n.

3. Solve the following recurrence relations, given that, for each, $a_0 = a_1 = 0$.

(i) $a_n - 4a_{n-1} + 3a_{n-2} = 7 \qquad (n \geq 2)$;

(i) $a_n - 7a_{n-1} + 12a_{n-2} = 7 \qquad (n \geq 2)$.

4. (a) Let $A_1, A_2, \ldots, A_n$ be finite sets, and let α_i denote the sum of the cardinalities of the intersections of the sets taken i at a time. Prove that

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} \alpha_i (-1)^{i+1}.$$

(b) How many integers are there in the range 1 to 999 (inclusive) which are not divisible by any of 2, 5 or 23?

(c) Let $S(n, k)$ denote the number of ways of partitioning an n-set into k parts. Write down a recurrence relation for $S(n, k)$ and use this relation to find all values of $S(n, k)$ for $1 \leq k \leq n \leq 5$.

(d) How many ways are there to distribute 5 distinguishable balls amongst 6 distinguishable boxes such that exactly three of the boxes are nonempty?

5. (a) [In this question, $p(n|\mathcal{P})$ denotes the number of partitions of the positive integer n having property $\mathcal{P}$.]

Write down the generating function for $p(n)$, the number of partitions of n.

Let $k \geq 1$ be fixed, let $\mathcal{P}_1$ be the property "No part in the partition appears more than k times" and let $\mathcal{P}_2$ be the property "No part in the partition is divisible by $k + 1$". Prove that $p(n|\mathcal{P}_1) = p(n|\mathcal{P}_2)$.

(b) Show that the number of partitions of n is equal to the number of partitions of $2n$ with the largest part equal to n.

Show also that the number of partitions of n is equal to the number of partitions of $2n$ into exactly n parts.