Section B

4. (a) Give a sketch of the proof of Frucht's Theorem:

> If Γ is a finite group, then there exists a graph G such that $\mathrm{Aut}(G) \simeq \Gamma$.

(b) Let A_n be the alternating group acting on the set $N = \{1, 2, \ldots, n\}$ with $n > 2$. Show that there is no graph G with $V(G) = N$ and such that the action of the permutation group $\mathrm{Aut}(G)$ on $V(G)$ is equivalent to that of A_n.

(c) Give a sketch of the proof of Bouwer's Theorem:

> Let Γ be a group of permutations acting on the set X. Then there exists a graph G such that $X \subseteq V(G)$, $\mathrm{Aut}(G) \simeq \Gamma$, X is invariant under the action of $\mathrm{Aut}(G)$, and the action of $\mathrm{Aut}(G)$ on X is equivalent to that of Γ.

(d) It is required to construct two graphs G_1, G_2 with the following properties. Both graphs have endvertices; the set of endvertices in G_i is denoted by X_i. Moreover,

- The endvertices of G_1 are all mutually pseudosimilar.
- The graph G_2 is not reconstructible from the collection of subgraphs $G_2 - v, v \in X_2$.

For ***only one*** of the two graphs G_1, G_2, explain how, with the aid of Bouwer's Theorem, the graph G_i can be constructed if it is possible to construct a permutation group Γ_i acting on X_i in a specified manner. [You should state clearly what conditions the action of Γ_i on X_i must satisfy, but you do not need to construct a specific example of a permutation group with these conditions.]

5. (a) Show that if H is a Cayley graph then there is a subgroup $\mathcal{H}$ of $\mathrm{Aut}(H)$ that acts regularly on $V(H)$.

Now let G be a graph on at least five vertices that is edge-transitive but not bipartite and whose vertices have odd degree. Let $H = L(G)$ be the line-graph of G. Show that H is a vertex-transitive graph that is not a Cayley graph.

[In the proof you may use the relevant theorems of Whitney and Tutte without proving them, but their use must be clearly indicated.]

(b) Let Γ be a finite group and let a, b be two distinct elements of Γ of order 3 and such that $\Gamma = \langle a, b \rangle$. Consider the Cayley graph $G = \mathrm{Cay}(\Gamma, S)$, where $S = \{a, b, a^{-1}, b^{-1}\}$. Show that any two different 3-cycles in G have at most a common vertex.
[HINT: Start off by assuming, for contradiction, that there are two 3-cycles with a common edge and that, without loss of generality, this common edge is $\{1, a\}$.]

6. Let G be a graph on n vertices and $m \geq n$ edges. In the sequel, for any graph H, $\binom{G}{H}$ denotes the number of subgraphs of G isomorphic to H. Let C_t denote the cycle on t vertices. Prove that

$$\binom{G}{C_n} = \binom{m}{n} - \sum_{t=3}^{n-1} \binom{G}{C_t} \cdot \binom{m-t}{n-t} + \sum_X b_X \cdot \binom{G}{X},$$

where the last summation ranges over all isomorphism types X of graphs with n edges, no isolated vertices, and containing at least two cycles, and where $b_X = (\text{number of cycles in } X) - 1$.

Deduce that $\binom{G}{C_n}$ is reconstructible from the deck of vertex-deleted subgraphs of G.

Indicate briefly how this result gives that the characteristic polynomial of G is reconstructible.

[Kelly's Lemma may be used without proof.]