MA271—June 1994

1. The sequence $< a_n >$ is defined by $a_0 = 1, a_1 = 0$ and the recurrence relation

$$na_n + a_{n-1} + a_{n-2} = f(n) \qquad (n \geq 2)$$

where the function f is given by $f(2n) = 0$ and $f(2n+1) = (-1)^n/(n!2^n)$.

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for this sequence. Prove that

$$g'(x) + (1+x)g(x) = e^{-x^2/2}.$$

Solve this differential equation and hence find a_n.

2. In this question, $p(n|\mathcal{P})$ denotes the number of partitions of the positive integer n having property $\mathcal{P}$.

(i) Show that

$$\begin{aligned} & p(n|\,\text{largest part} = m \text{ and number of parts} = k) \\ = \; & p(n|\,\text{largest part} = k \text{ and number of parts} = m). \end{aligned}$$

Deduce that

$$\begin{aligned} & p(n-m|\,\text{number of parts} = k-1, \text{ no part greater than } m) \\ = \; & p(n-k|\,\text{number of parts} = m-1, \text{ no part greater than } k). \end{aligned}$$

(ii) Show that

$$\begin{aligned} & p(n|\,\text{no part occurs more than } 2^{k+1}-1 \text{ times}) \\ = \; & p(n|\,\text{no multiple of } 2^k \text{ occurs more than once as a part}). \end{aligned}$$

(iii) Let

$$e_n = p(n|\,\text{even number of parts, all distinct})$$

and

$$o_n = p(n|\,\text{odd number of parts, all distinct}).$$

Write down an expression for $e_n - o_n$ and show how this relationship can be used to obtain a recursive method for computing $p(n)$, the number of partitions of n.

3. (a) Suppose $A_1, A_2, \ldots, A_n$ are finite sets. Show that

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} (-1)^{i+1} \alpha_i$$

where α_i denotes the sum of the cardinalities of all the intersections of the sets taken i at a time.

(b) A box contains n white balls and m black balls. A set of k balls is to be chosen from the box. Let the white balls be $w_1, w_2, \ldots, w_n$ and let A_j be the set of all sets of k balls which contain w_j. By counting in two ways the number of sets of k *black* balls which can be chosen, prove that

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{m+n-i}{k-i} = \begin{cases} \binom{m}{k} & \text{if } m \geq k \\ 0 & \text{if } m < k \end{cases}$$

(c) Let $S(n,k)$ denote the number of partitions of an n-set into k parts. Show that the number of surjections from an n-set to a k-set is $k!S(n,k)$. Show that

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$