Combinatorics 2—June 1994

1. Let D be a finite set and Γ a group of permutations acting on D. A *structure* is defined to be a colouring of the elements of D with the colours $\{0,1\}$. Let R, T be two structures and X a subset of D coloured 1 in R. Explain what is meant by $|R \to T|$ and $|R \xrightarrow{X} T|$, and write down, without proof, an expression for $|R \xrightarrow{X} T|$ in terms of $|S \to T|$ for substructures S of R, explaining clearly the notation used.

Define the term *deck* of a structure.

Suppose R and T have the same deck. Prove that

$$|R \to T| = |R \to R| + (-1)^{|X|}(|R \xrightarrow{X} T| - |R \xrightarrow{X} R|.$$

[Kelly's Lemma may here be assumed.]

Hence show that, if R is not uniquely reconstructible from its deck and $|X|$ is even, then $|R \xrightarrow{X} R| > 0$. Deduce that, if R is not reconstructible, then

$$2^{m-1} \leq |\Gamma|$$

where m is the number of elements coloured 1 in R.

(i) Equally spaced lines are drawn across the faces of a cube so that each face is an $n \times n$ grid, and m of the cells in these grids are coloured black, the rest coloured white. The deck of the cube is the family of all cubes obtained by changing, one at a time, the colour of each black cell into white. Show that, if $m \geq 6$, then the cube is reconstructible from its deck.

(ii) Let G be a graph on n ($n \geq 10$) vertices and let H be a graph on 10 vertices and 25 edges. Suppose that G does not contain any induced subgraph isomorphic to H but G contains a subgraph isomorphic to the complete graph on 10 vertices. Prove that G is (edge) reconstructible.

2. Let $r = (123\ldots n)$ be the cyclic permutation of the set $X = \{1, 2, \ldots, n\}$. Show that the cycles of r^i are all of the same length, and hence obtain the cycle index of the cyclic group C_n of permutations of A generated by r.

A circular disc is divided into n equal sectors on one side, and each sector is painted white or black. Show that the number of discs which can be made this way having exactly k black sectors is

$$\frac{1}{n} \sum_{d|(k,n)} \phi(d) \binom{n/d}{k/d}$$

where (k, n) is the greatest common divisor of k and n, and ϕ is Euler's ϕ-function.

3. Let G be a group of permutations acting on a finite set D. For any $g \in G$, let $F(g) = \{x \in D : g(x) = x\}$. Prove that the number of orbits of D under the action of G is given by

$$\frac{1}{|G|} \sum_{g \in G} |F(g)|.$$

[The Orbit-Stabiliser Theorem may be used without proof.]

Identity cards are to be manufactured from plastic squares marked with an $n \times n$ grid (n odd) on both sides and with two holes punched into them (one hole through each of two cells of the grid). Show that the number of distinguishable identity cards which can be manufactured this way is

$$\frac{(n-1)(n^3+n^2+9n+1)}{16}.$$

4. (a) Let p, q be positive integers and let $r(p, q)$ denote the least value of n such that any 2-colouring of the edges of the complete graph K_n contains a monochromatic K_p or K_q as a subgraph. Show that

$$r(p, q) \leq \binom{p+q-2}{p-1}.$$

(b) A *tournament* is an orientation of the edges of a complete graph. A tournament is said to be *transitive* it its vertices can be labelled $1, 2 \ldots, k$ such that an arc joins i *to* j if and only if $i < j$. Show that if $k < \log_2 n$ then any tournament on n vertices has a transitive subtournament on k vertices.

[Hint: Fix a vertex v_0 and consider the two sets of vertices incident to, respectively from, v_0.]