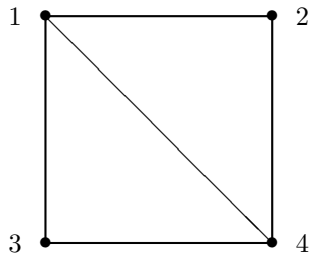**Section B: Combinatorics**

1. Find the cycle index of the automorphism group of the graph shown below, acting on the set of vertices of the graph.

Suppose the vertices of the above graph are to be coloured and the four colours red, gree, blue and yellow are available. (The colourings here need not be "proper", that is, adjacent vertices can be given the same colour.)

Use Polya's Theorem to find the number of colourings of the graph which are non-equivalent under the action of its automorphism group and in which,

(i) exactly two vertices are coloured red and the others are given two different colours;

(ii) all vertices are given different colours;

(iii) only two colours are used, and two vertices are given one colour while the other two are given the other colour.

2. If G is a group of permutations on $N = \{1, 2, \ldots, n\}$, let $G^{(2)}$ denote the group of actions induced by G on unordered pairs of N.

Now consider the group S_4, the symmetric group on $\{1, 2, 3, 4\}$. Find the cycle index of S_4 and of $S_4^{(2)}$.

Hence write down the generating function for the number of nonisomorphic graphs on four vertices and m edges, $0 \le m \le 6$.

[*Polya's Theorem may be used without proof.*]

3. Let p, q be integers and let $r(p, q)$ denote the least value of n such that any colouring of the edges of the complete graph K_n with colours red or blue contains a red K_p or a blue K_q. Prove that

$$r(p, p) \ge 2^{p/2}.$$

Show also that if $n \leq 2^{p/2}$ and the edges of K_n are coloured red or blue at random (such that all possible colourings are equally likely) then the probability of a monochromatic triangle is less than $2.2^{p/2}/p!$.

Suppose that the edges of K_9 are coloured red or blue, and let v be a vertex of K_9. Show that if v is incident to four red edges then K_9 either contains a red K_3 or a blue K_4. Show that the same conclusion holds if v is incident to six blue edges. [*You may assume here that* $r(3,3) = 6$.]

Show that it is not possible that every vertex of K_9 is incident to exactly three red and five blue edges. [*Hint: Use the Handshaking Lemma.*]

Deduce that $r(3,4) \leq 9$.

4. Suppose C is a q-ary code of length n, and suppose C is e-error-correcting. What value must the minimum distance δ in C at least have?

Prove that
$$|C| \leq \frac{q^n}{\sum_{i=0}^{e} \binom{n}{i}(q-1)^i}.$$

Now let C be a linear code. What is the minimum nonzero weight in C?

Let H be a check matrix for C. Prove that any e columns of H are linearly independent.

Now let C be the binary Hamming code $H(r, 2)$ which is the code with check matrix H whose columns are all the distinct non-zero vectors in $\{0, 1\}^r$. What is the length and the dimension of C?

Show that C is a perfect 1-error-correcting code.

Let C_1 be the above Hamming code with $r = 3$. Suppose the probability of an error in the transmission of any digit is p (independently of the other digits), and suppose a transmitted word is decoded adopting the nearest neighbour procedure. What is the probability (in terms of p) that a received word is decoded correctly?

Now suppose C_2 is the 3-fold repetition code of length 12 and information rate 1/3. Show that the sizes of C_1 and C_2 are the same. Find, in terms of p, the probability that a received word is decoded correctly adopting the nearest neighbour procedure with code C_2.