UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc. (Hons.) II Year
June Session 2002

MA112 Groups (1.5 credits) **20 June 2002**

1415–1615

Answer THREE questions

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1. Let G be a finite group acting on a finite set X. For $x \in X$ let $G(x)$ and G_x denote, respectively, the orbit and the stabiliser of x. Prove that

$$|G| = |G(x)| \cdot |G_x|.$$

Now suppose $|X| \leq 90$ and suppose G is a 7-group acting on X and having exactly one fixed point. Suppose also that H is an 11-group acting on X and that the action of H has no fixed points. Find $|X|$.

2. (a) Let G be a finite group, $H \leq G$ and X the set of left cosets of H in G. Show that there is an action of G on X such that the kernel of this action is contained in H.

Suppose G is a group of order 70 and suppose also that G contains a subgroup of order 14. Show that G cannot be simple.

(b) State carefully the three Sylow Theorems.

Prove that a group of order 992 cannot be simple.

3. Let G be a finite group acting on a finite set X. For each $g \in G$, let $F(g)$ denote the set $\{x \in X : \hat{g}(x) = x\}$, where $\hat{g}$ denotes the permutation of X corresponding to g under the action.

Prove that the number of orbits in X under this action is given by

$$\frac{1}{|G|} \sum_{g \in G} |F(g)|.$$

[The Orbit-Stabiliser Theorem may be assumed without proof.]

A necklace is to be made from 9 beads strung on a circular wire; 6 of these beads are to be coloured white and 3 beads are to be coloured black. Ignoring the positioning of the fastening, how many essentially different necklaces can be made this way?

4. Obtain the class equation for a finite group, explaining clearly the terms *conjugacy, centre* and *conjugacy class*. Explain also why the order of a conjugacy class divides the order of the group.

Let G be a group of order 24 with centre consisting only of the identity element. Show that G has a conjugacy class of size 3, and deduce that G has a subgroup of order 8.

[You may use the Orbit-Stabiliser Theorem in this question, but Sylow's Theorems may **not** be used.]