UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc. (Hons.) II Year
June Session 2003

MAT2104 Groups (2 credits) **2 June 2003**

0915–1145

Answer FOUR questions

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1. Let G be a group and H a subgroup of G. Let $[G : H] = m$ and let X be the set of all left cosets of H in G.

Show that there is an action of G on X whose kernel K is contained in H.

Show also that

$$\frac{|G|}{|K|} \text{ divides } m!.$$

Suppose that $|G| = 1001$ and $|H| = 143$. Prove that H is a normal subgroup of G.
[If required, the First Isomorphism Theorem may be quoted without proof.]

2. Let G be a finite group. Write down the class equation for G, explaining clearly all the terms involved and the associated group action.

In the sequel let G be a p-group. Prove that the centre $Z(G)$ of G is nontrivial. Show that if G acts on a set X, and X_1 is the set of fixed points of this action, then

$$|X_1| = |X| \bmod p.$$

Deduce that if H is a nontrivial normal subgroup of G then $|H \cap Z(G)| \geq p$.
[*Hint.* For the last part, consider the action of conjugacy on $X = H$.]

[The Orbit-Stabiliser Theorem may be quoted without proof.]

3. State Sylow's Theorems.

Show that a group of order 56 cannot be simple.

Let G be a group of order 110. How many elements of order 11 are there in G? What are the possible numbers of elements of order 5 in G? Give two examples of groups of order 110 having different numbers of elements of order 2.

4. Let G be a finite group acting on a finite set X. For each $g \in G$, let $F(g)$ denote the set $\{x \in X : \hat{g}(x) = x\}$, where $\hat{g}$ denotes the permutation of X corresponding to g under the action.

Prove that the number of orbits in X under this action is given by

$$\frac{1}{|G|} \sum_{g \in G} |F(g)|.$$

[The Orbit-Stabiliser Theorem may be assumed without proof.]

A necklace is to be made from 9 beads strung on a circular wire; 6 of these beads are to be coloured white and 3 beads are to be coloured black. Ignoring the positioning of the fastening, how many essentially different necklaces can be made this way?

5. (a) Let S_n be the group of all permutations of the set $\{1, 2, \ldots, n\}$. For $\alpha \in S_n$, let $c(\alpha)$ denote the number of cycles of α when it is written as a product of disjoint cycles. Let $\beta \in S_n$ be a transposition. Give, with proof, the possible values of $c(\beta\alpha)$ in terms of $c(\alpha)$. Deduce from this that if α is written as a product of transpositions then the number of transpositions is always either odd or even.

(b) Let G be a subgroup of S_n, and let H be the subgroup of G consisting of all the even permutations in G. Show that either $H = G$ or else $|H| = |G|/2$. (*Hint:* Define a suitable surjective homomorphism from G to $\{1, -1\}$ and find the kernel of this homomorphism. You may use the First Isomorphism Theorem, if required.)

Deduce that A_n, the group of all even permutations in S_n, has order equal to $|S_n|/2$.