UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc. (Hons.) B.Ed. (Hons.) II Year
June Session 2004

MAT2104 Groups (4 ECTS credits) **12 June 2004**

Time allowed: 2 hours 30 minutes — 0900–1130

Answer FOUR questions

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1. Let G be a finite group acting on a finite set X. For $x \in X$ let $G(x)$ and G_x denote, respectively, the orbit and the stabiliser of x. Prove that if x and y are in the same orbit and $g, h \in G$ such that $\hat{g}(x) = \hat{h}(x) = y$ (where $\hat{g}$ denotes the permutation of X corresponding to g under the action), then the two left cosets gG_x and hG_x are equal. Deduce that

$$|G| = |G(x)| \cdot |G_x|.$$

Without using any of Sylow's Theorems, prove that if the prime number p divides the order of G then G contains an element of order p.

2. Let G be a finite group. Write down the class equation for G, explaining clearly all the terms involved and the associated group action.

In the sequel let G be a p-group. Prove that the centre $Z(G)$ of G is nontrivial. Show that if G acts on a set X, and X_1 is the set of fixed points of this action, then

$$|X_1| = |X| \bmod p.$$

Deduce that if H is a nontrivial normal subgroup of G then $|H \cap Z(G)| \geq p$. [*Hint.* For the last part, consider the action of conjugacy on $X = H$.]

[The Orbit-Stabiliser Theorem may be quoted without proof.]

3. Let G be a finite group acting on a finite set X. For each $g \in G$, let $F(g)$ denote the set $\{x \in X : \hat{g}(x) = x\}$, where $\hat{g}$ denotes the permutation of X corresponding to g under the action.

Prove that the number of orbits in X under this action is given by

$$\frac{1}{|G|} \sum_{g \in G} |F(g)|.$$

[The Orbit-Stabiliser Theorem may be quoted without proof.]

A 10×10 square grid is drawn on one side of a cardboard square. Four identical circles are drawn, each circle inside one of the cells of the grid. How many different cards can be drawn this way?

4. (a) You are given this information about the group G: G is not cyclic and it contains a subgroup H of order p (p an odd prime) and index 2. Describe fully the structure of G, explaining clearly your reasoning.

(b) Classify all groups of order 45.

[In (b) you may use without proof any of the Sylow Theorems and any results on p-groups and direct products. Any of these results which you do use should, however, be clearly stated.]

5. Let G be a group, H a subgroup of G, and X the set of left cosets of H in G. Show that there is an action of G on X such that the kernel of the action is a normal subgroup of G which is contained in H.

Deduce that if $|G| = p^n$ and $|H| = p^{n-1}$ (p a prime), then H is a normal subgroup of G.

[*Hint for second part:* Let θ be the action defined in the first part, let $\ker(\theta)$ be its kernel, and let $\theta(G)$ be its range. From the First Isomorphism Theorem, deduce that $|\theta(G)| = p^k$, for some k. From $\theta(G) \leq S_X$ deduce that $k \leq 1$. From the First Isomorphism Theorem again deduce that $|\ker(\theta)| \geq p^{n-1}$.]