UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc.(Hons.) Year IV
June 2002 Examination Session

Mathematics Elective Paper I **21 June 2002**

0900–1200

Answer FOUR questions: TWO questions from each section

Section A: MA343 Graph Theory and Combinatorics I

1. (a) Let p be a positive integer, and let $r(p,p)$ be the least value of n such that any colouring of the edges of the complete graph K_n with colours red and blue contains either a red or a blue K_p. Show that

$$r(p,p) > 2^{p/2}.$$

(b) Let P_3 be the path on three edges and C_4 the cycle on four edges, and let $r(P_3, C_4)$ denote the least value of n such that any colouring of the edges of the complete graph K_n with colours red and blue contains either a red P_3 or a blue C_4. Show that

$$r(P_3, C_4) = 5.$$

2. If G is the group of permutations of $N = \{1, 2, \ldots, n\}$, then $G^{[2]}$ denotes the group of actions induced on ordered distinct pairs of N; that is, to each $g \in G$ there exists $\hat{g} \in G^{[2]}$ such that $\hat{g}((i,j)) = (g(i), g(j))$.

Suppose G is S_4, that is, the group of all permutations of $\{1, 2, 3, 4\}$. Find the cycle index of $G^{[2]}$ given that the cycle index of G is

$$\frac{1}{24}(x_1^4 + 8x_1x_3 + 6x_1^2x_2 + 3x_2^2 + 6x_4).$$

Hence obtain the number of nonisomorphic digraphs on four vertices.
[“Burnside’s” Counting Theorem may be used without proof.]

3. Let $(X, \mathcal{B})$ be a 2-design with $|X| = v$, and b blocks of size k each. Let λ be the number of blocks containing any given pair of elements and suppose that each element is contained in exactly r blocks. Show that

$$bk = vr$$

and

$$r(k-1) = \lambda(v-1).$$

Obtain also Fisher's inequality, $v \leq b$. [You may assume that the design is non-trivial, that is, $r > \lambda$.]

Deduce that there cannot exist a design with $v = 25$, $k = 10$ and $\lambda = 3$.

4. (a) The code C consists of a set of words of length n taken from the alphabet $\mathbb{Z}_2$. Suppose that all codewords are equally likely to be transmitted and that each digit has a probability $p < 1/2$ of being transmitted wrongly, independently of the other digits. A transmitted word will be decoded either as the codeword nearest (in Hamming distance) to the one received (*nearest neighbour decoding*) or as the word most likely to have been transmitted, given the one received (*most likelihood decoding*). Show that these two methods are equivalent under the above assumptions.

Comment briefly on how reasonable in practice these assumptions are.

(b) Show that a code is e-error-correcting if and only if the minimum distance between any two codewords is at least $2e+1$.

(c) Suppose that the code C consists of a set of words of length n taken from an alphabet F of size q and that the minimum distance between any two codewords is δ. Show that

$$|C| \geq \frac{q^n}{\sum_{i=0}^{\delta-1} \binom{n}{i}(q-1)^i}$$

and that if C is e-error-correcting then

$$|C| \leq \frac{q^n}{\sum_{i=0}^{e} \binom{n}{i}(q-1)^i}$$