UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc.(Hons.) Year IV
June 2001 Examination Session

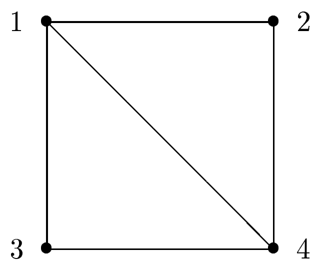
Mathematics Elective Paper I **June 2001**

0915–1215

Answer FIVE questions, attempting ONE question from each of FIVE sections.

Section A: MA341 Combinatorics

1. Find the cycle index of the automorphism group of the graph shown below, acting on the set of vertices of the graph.



Suppose the vertices of the above graph are to be coloured and that four colours *red, green, blue* and *yellow* are available. (The colourings here need not be 'proper', so adjacent vertices can be given the same colour.)

Use Pólya's Theorem to find the number of colourings of the graph which are non-equivalent under the action of its automorphism group and in which,

(i) exactly two vertices are coloured red and the others are given two different colours;

(ii) all vertices are given different colours;

(iii) only two colours are used, and two vertices are given one colour while the other two are given the other colour.

2. (a) Let p be a positive integer, and let $r(p)$ be the least value of n such that any colouring of the edges of the complete graph K_n with colours red and blue contains either a red or a blue K_p. Show that

$$r(p) > 2^{p/2}.$$

(b) Let S_k be the symmetric group acting on a set A with k elements, and let $p(n|k)$ be the number of partitions of n with exactly k parts. Suppose that the vertices of the complete graph K_k on k vertices are coloured with the integers $1, 2, 3, \ldots$. By considering the number of non-equivalent ways in which K_k can be coloured such that the sum of the colours on its vertices gives a total of n, and using Pólya's Theorem, show that $p(n|k)$ is the coefficient of c^k in

$$P_{S_k}(\frac{c}{1-c}, \frac{c^2}{1-c^2}, \frac{c^3}{1-c^3} \ldots)$$

where $P_{S_k}(x_1, x_2, \ldots, x_k)$ is the cycle index of S_k.