UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc. (Hons.) B.Ed. (Hons.) II Year
May/June Session 2004

MAT2104 Groups (4 ECTS credits) **May 2005**

Time allowed: 2 hours 30 minutes

Answer FOUR questions

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1. Let G be a finite group acting on a finite set X. For $x \in X$ let $G(x)$ and G_x denote, respectively, the orbit and the stabiliser of x. Prove that if x and y are in the same orbit and $g, h \in G$ such that $\hat{g}(x) = \hat{h}(x) = y$ (where $\hat{g}$ denotes the permutation of X corresponding to g under the action), then the two left cosets gG_x and hG_x are equal. Deduce that

$$|G| = |G(x)| \cdot |G_x|.$$

Now, suppose that the action is transitive, that is, all the elements of X are in one orbit. For any $x \in X$ let $m(x)$ be the number of permutations under this action which do *not* fix x. Show that

$$\sum_{x \in X} m(x) = |G| \cdot (|X| - 1).$$

[*Hint.* The permutations which do not fix x are those in $G - G_x$.]

2. Let G be a finite group. Write down the class equation for G, explaining clearly all the terms involved and the associated group action.

In the sequel let G be a p-group. Prove that the centre $Z(G)$ of G is nontrivial. Show that if G acts on a set X, and X_1 is the set of fixed points of this action, then

$$|X_1| = |X| \bmod p.$$

Deduce that if H is a nontrivial normal subgroup of G then $|H \cap Z(G)| \geq p$.
[*Hint.* For the last part, consider the action of conjugacy on $X = H$.]

[The Orbit-Stabiliser Theorem may be quoted without proof.]

3. Let G be a finite group acting on a finite set X. For each $g \in G$, let $F(g)$ denote the set $\{x \in X : \hat{g}(x) = x\}$, where $\hat{g}$ denotes the permutation of X corresponding to g under the action.

Prove that the number of orbits in X under this action is given by

$$\frac{1}{|G|} \sum_{g \in G} |F(g)|.$$

[The Orbit-Stabiliser Theorem may be quoted without proof.]

Identity cards are to be made this way. A 10×10 square grid is drawn on one side of a cardboard square. Four identical circles are drawn, each circle inside one of the cells of the grid. How many different cards can be drawn this way?

4. (a) State the three Sylow Theorems and prove one of them. [The Orbit-Stabiliser Theorem may be quoted without proof.]

(b) Prove that a group of order 992 cannot be simple.

5. Let G be a group and H a subgroup of G. Let $[G : H] = m$ and let X be the set of all left cosets of H in G.

Show that there is an action of G on X whose kernel K is contained in H.

Show also that

$$\frac{|G|}{|K|} \text{ divides } m!.$$

[The First Isomorphism Theorem may be quoted without proof.]

Suppose that $|G| = 1001$ and $|H| = 143$. Prove that H is a normal subgroup of G.
[Consider the possible values of K and use the previous result.]