UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc.(Hons.) Year IV
June 2004 Examination Session

Mathematics Elective Paper II — MAT4403 **14 June 2004**

0900–1200

Answer FIVE questions with at least TWO questions from each section

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Section A: Graph Theory and Combinatorics I

1. (a) Let C be a q-ary code of length n and minimum Hamming distance δ. Prove the Singleton Bound, namely that

$$|C| \leq q^{n-\delta+1}.$$

(b) Let C be a *binary linear* code of length n. Suppose also that C is to be used only for error-detection, that is, if a word is received which is not in C then the receiver asks for retransmission. Therefore an error is undetected if and only if the received word $\mathbf{y}$ is a codeword different from the codeword $\mathbf{x}$ which was sent.

Why is it that the error pattern $\mathbf{e}$ in this case is itself a non-zero codeword?

Let A_i denote the number of codewords in C having weight i. Suppose that any digit of a codeword can be sent incorrectly with probability p, independently of the other digits. Deduce that the probability of an incorrect message being received undetected is given by

$$P_{undetect}(C) = \sum_{i=1}^{n} A_i p^i (1-p)^{n-i}.$$

(c) Let the code C in (b) above have generator matrix

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Find C and show that

$$P_{undetect}(C) = p^2 - p^4.$$

2. (a) Let $(X, \mathcal{B})$ be a 2-design for which $|X| = v$, the size of each block is k, and every 2-set of X appears in exactly λ blocks.

Let $x_0 \in X$ and suppose that x_0 occurs in exactly r blocks. By counting in two ways the number of pairs $\{x_0, y\}$, for all $y \in X$, which appear together in the same block, show that

$$r(k-1) = \lambda(v-1).$$

Deduce that r is independent of the element x.

Assuming that the design has b blocks, show that $bk = vr$ and $\lambda \le r$. (For the inequality, compare k and v.)

(b) Find the non-zero squares in the field $\mathbf{Z}_{11}$ and show that they form a difference set. Use this set to construct a symmetric design, and give the parameters of this design.

3. (a) Suppose that the edges of an infinite complete graph G are coloured red or blue. Show that G contains an infinite complete subgraph whose edges are all of the same colour.

Deduce that an infinite sequence of real numbers contains an infinite monotonic subsequence.

(b) Marbles of $n+1$ different colours are placed in n jars. There are $n+1$ marbles of each colour.

Show that, for each colour, there is a jar containing a pair of marbles of that colour. Deduce that there is a jar containing two pairs of marbles from two different colours.

4. (a) Let $s = (1\ 2\ 3 \ldots\ n)$ be the cyclic permutation of the set $D = \{1, 2, \ldots, n\}$. Show that the cycles of s^i are all of the same length. Hence obtain, in terms of Euler's ϕ-function, the cycle index of the cyclic group of permutations of D generated by s.

(b) Let p be a prime number and let r and t be positive integers. Prove that p^t divides

$$r^{p^t} + \sum_{i=1}^{t} p^{i-1}(p-1)r^{p^{t-i}}.$$

[*Hint.* Any form of "Burnside's" Counting Lemma may be used without proof, but its use must be clearly indicated.]