UNIVERSITY OF MALTA
FACULTY OF SCIENCE
Department of Mathematics
B.Sc.(Hons.) Year IV
June 2005 Examination Session

Mathematics Elective Paper II — MAT4403 **June 2005**

0900–1200

Answer FIVE questions with at least TWO questions from each section

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Section A: Graph Theory and Combinatorics I

1. (a) Let $r(p)$ denote the smallest value of N such that any 2-colouring of the edges of the complete graph K_N must contain a monochromatic K_p. Show that

$$r(p) > 2^{p/2}.$$

(b) Let r_n denote the generalised Ramsey number $r(p_1, p_2, \dots, p_n)$ with $p_i = 3$ for all i, that is, r_n denotes the smallest value of N such that any colouring of the edges of K_N with n colours must contain a monochromatic triangle. Show that

$$r_n \leq n(r_{n-1} - 1) + 2$$

and deduce that

$$r_n \leq \lfloor n!e \rfloor + 1.$$

[Recall that $r_2 = 6$.]

2. (a) Let $(X, \mathcal{B})$ be a 2-design with $|X| = v$, and b blocks of size k each. Let r be the number of blocks containing any single element and let λ be the number of blocks containing any given pair of elements. Prove that:

(i) $bk = vr$;

(ii) $r(k-1) = \lambda(v-1)$.

(b) Let C be a linear, perfect e-error-correcting binary code of length n. Let $\mathcal{B}$ be the collection of all codewords of weight $2e+1$. Let each codeword c in $\mathcal{B}$ correspond to the subset A of $\{1, 2, \dots, n\}$ defined by $i \in A$ if and only if $c_i = 1$.

Show that this family of subsets gives an $(e+1)$-design with $\lambda_{e+1} = 1$. [*Hint.* The spheres $\{S_e(c) : c \in C\}$ partition $\mathbb{Z}_2^n$ since C is perfect.]

3. If G is the group of permutations of $N = \{1, 2, \ldots, n\}$, then let $G^{[2]}$ denote the group of actions induced on ordered distinct pairs of N; that is, to each $g \in G$ there exists $\hat{g} \in G^{[2]}$ such that $\hat{g}((i,j)) = (g(i), g(j))$.

Suppose G is S_4, that is, the group of all permutations of $\{1, 2, 3, 4\}$. Find the cycle index of $G^{[2]}$ given that the cycle index of G is

$$\frac{1}{24}(x_1^4 + 8x_1x_3 + 6x_1^2x_2 + 3x_2^2 + 6x_4).$$

Hence obtain the number of nonisomorphic digraphs on four vertices.

["Burnside's" Counting Theorem may be used without proof.]

4. (a) Let C be a q-ary code of length n and minimum Hamming distance δ. Prove the Singleton Bound, namely that

$$|C| \leq q^{n-\delta+1}.$$

Now let C be a linear code and let ω be the minimum weight amongst all nonzero codewords in C. Show that

$$\omega = \delta.$$

(b) Suppose that the linear code C is constructed as follows. A finite field $F = \{u_1, u_2 \ldots, u_q\}$ is given and S is the space of all polynomials of degree at most $k-1$ with coefficients in F. The code C consists of all codewords

$$(p(u_1), p(u_2), \ldots, p(u_q))$$

for all p in S.

Find the minimum distance in C (you may assume that any p in S has at most $k-1$ zeros) and the dimension of C (use a basis for S; you may assume that

$$\det \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & u_1 & u_2 & \ldots & u_{k-1} \\ 1 & u_1^2 & u_2^2 & \ldots & u_{k-1}^2 \\ \vdots & & & & \\ 1 & u_1^{k-1} & u_2^{k-1} & \ldots & u_{k-1}^{k-1} \end{pmatrix}$$

is not zero.)

Show that C attains the Singleton Bound.