Section B

4. (a) Show that a graph G is a Cayley graph $\mathrm{Cay}(\Gamma, S)$ if $(\mathrm{Aut}(G), V(G))$ contains a sub-permutation group isomorphic to Γ acting regularly on $V(G)$. Deduce that if G is a graphical regular representation (GRR) of its automorphsim group Γ then G is isomorphic to some Cayley graph $\mathrm{Cay}(\Gamma, S)$. [11]

(b) Show how a GRR of a group of odd order can be used to construct a graph each of whose vertices has a pseudosimilar mate. [11]

(c) Call a vertex u of G *irreplaceable* if for no set of neighbours A of u is there a set B of vertices of G not adjacent to u such that G is isomorphic to the graph obtained by removing all edges in $\{ua : a \in A\}$ and replacing them by edges $\{ub : b \in B\}$.

Show that an irreplaceable vertex u of G cannot be pseudosimilar to any vertex to which it is not adjacent. [12]

5. Let Γ be a group, $S \subseteq \Gamma$ such that $S^{-1} = S$, and G the Cayley graph $\mathrm{Cay}(\Gamma, S)$.

(a) Let ϕ be an automorphism of the group Γ such that $\phi(S) = S$. Show that ϕ is an automorphism of the graph G fixing the vertex 1. [8]

(b) Suppose that G is a (GRR) of the group Γ. Show that if Γ is abelian, then it is an elementary abelian 2-group. [12]

(c) Let Γ be the elementary abelian 2-group $\mathbb{Z}_2^5$ generated by the distinct elements $a_i, 1 \le i \le 5$. Let S be the set

$$S = \{a_i, a_k a_{k+1}, a_1 a_2 a_3 a_4, a_1 a_2 a_4 a_5 : 1 \le i \le 5, 1 \le k < 5\}.$$

Consider the Cayley graph $G = \mathrm{Cay}(\Gamma, S)$, and suppose ϕ is an automorphism which fixes 1. Let H be the subgraph of G induced by the neighbours of 1. Show that H has the trivial automorphism group and deduce that G is a GRR of Γ. [14]

6. (a) Let G be a graph without isolated vertices. Show that the deck of G is uniquely determined from the edge-deck of G. [Any form of Kelly's Lemma, if required, may be quoted without proof.] [13]

(b) Let G be a graph on at least three vertices. Prove that each of the following is reconstructible from the deck $\mathcal{D}(G)$:

(i) The number of edges of G;

(ii) For any $G - v \in \mathcal{D}(G)$, the degree in G of the missing vertex v;

(iii) For any $G - v \in \mathcal{D}(G)$, the degrees of the neighbours of v in G.

[10]

(c) Assuming that the minimum degree δ of a graph G is reconstructible from its edge-deck, show that a graph G is edge-reconstructible in each of the following cases.

(i) G contains two adjacent δ-vertices.

(ii) G contains a $(\delta + 1)$-vertex adjacent to two δ-vertices.

(iii) G contains a triangle with one δ-vertex and two $(\delta + 1)$-vertices.

[11]