Section B

4. Let G be a graph on n vertices and let A be the adjacency matrix of G. Let π be a permutation of $V(G)$ and P the permutation matrix representing π. Prove that π is an automorphism of G if and only if $PA = AP$. [7 marks]

Suppose, from now on, that π is an automorphism of G. Show that if x is an eigenvector of A corresponding to the eigenvalue λ, then Px is also an eigenvector of A corresponding to λ. [3 marks]

Deduce that if λ is a simple eigenvalue of A and x is a corresponding eigenvector with real components, then $Px = \pm x$. [7 marks]

Hence show that if all eigenvalues of G are simple, then every non-trivial automorphism of G has order 2. [7 marks]

Suppose now that the permutation π has s cycles of odd length and t cycles of even length, when written as a product of disjoint cycles. Prove that the number of simple eigenvalues of G is at most $s + 2t$. [10 marks]

5. (a) Show that if the automorphism group of a graph G contains a sub-permutation group Γ acting regularly on $V(G)$ then G is a Cayley graph of Γ with respect to some $S \subseteq \Gamma$ with $S = S^{-1}$. [10 marks]

(b) Let Γ be a finite group and $S \subseteq \Gamma$ a generating set of Γ with $S = S^{-1}$. Let G be the Cayley graph of Γ with respect to S. Show that if ϕ is an automorphism of Γ such that $\phi(S) = S$, then ϕ is also an automorphism of the graph G. [7 marks]

(c) Now let G be any graph, and let Γ =Aut(G). Suppose Γ is abelian and acts transitively on $V(G)$. Show that Γ is an elementary abelian 2-group. [You may assume that if an abelian subgroup of S_Y (the group of all permutations on Y) acts transitively on Y then it also acts regularly.] [7 marks]

(d) Let G be a connected cubic Cayley graph Cay(Γ, S) of an abelian group Γ, and suppose that G has order $2n$ which is greater than 8. Show that the elements of S are of the form $\alpha, \beta, \beta^{-1}$ where the order of α is 2 and that of β is greater than two. Show also that the order of β is either n or $2n$. [10 marks]

6. (a) Let G be a graph without isolated vertices. Show that the deck of G is uniquely determined from the edge-deck of G.
[Any form of Kelly's Lemma, if required, may be quoted without proof.] [12 marks]

(b) Assuming that the minimum degree δ of a graph G is reconstructible from its edge-deck, show that a graph G is edge-reconstructible in each of the following cases.

(i) G contains two adjacent vertices both of degree δ; [2 marks]

(ii) G contains a $(\delta + 1)$-vertex adjacent to two δ-vertices; [5 marks]

(iii) G contains a triangle with one δ-vertex and two $(\delta + 1)$-vertices. [3 marks]

(c) A graph is said to have property EA_k if $G - A \not\simeq G - B$ for any two distinct subsets A, B of $E(G)$ with $|A| = |B| = k$. Prove that if G has property EA_3 then it can be reconstructed from any two edge-deleted subgraphs in its edge-deck. [12 marks]