AN INVITATION TO TWO-FOLD ORBITAL DIGRAPHS: THE DISCONNECTED CASE

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NOTE!

All our graphs are digraphs in which the existence of any arc (x, y) implies the existence of the arc (y, x).

Together they form the edge $\{x, y\}$.

All digraphs are finite and without multiple arcs, but loops are allowed.

ORBITAL GRAPHS:

Let Γ be a permutation group acting transitively on a set V. Fix $(u, v) \in V \times V$. Then all pairs $(\alpha(u), \alpha(v))$, with $\alpha \in \Gamma$, form a digraph G, such that $\Gamma \leq Aut(G)$.

G is vertex- and arc-transitive.

If G is disconnected then all its components are isomorphic.

Notation:

$\mathcal{S}$ will denote $S_n \times S_n$.

$\mathbf{\Gamma}$ will denote a subgroup of $\mathcal{S}$.

Suppose $\pi_1, \pi_2 : \mathbf{\Gamma} \to S_n$ are defined by $\pi_1(\alpha, \beta) = \alpha$ and $\pi_2(\alpha, \beta) = \beta$. Then, π_1 and π_2 are said to be the *canonical projections* of $\mathbf{\Gamma}$ on S_n.

TWO-FOLD ORBITAL DIGRAPHS:

Let $\mathbf{\Gamma} \leq \mathcal{S}$ where $\pi_1\mathbf{\Gamma}$ and $\pi_2\mathbf{\Gamma}$ are transitive on the n-set V.

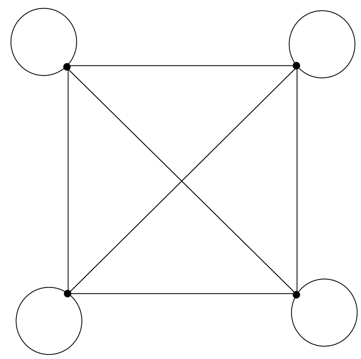
For a fixed element (u, v) in $V \times V$ let

$$\mathbf{\Gamma}(u, v) = \{(\alpha(u), \beta(v)) | (\alpha, \beta) \in \mathbf{\Gamma}\}$$

The set $\mathbf{\Gamma}(u, v)$ is called a *two-fold orbital*.

The digraph $G = (V, \mathbf{\Gamma}(u, v))$ is said to be a *two-fold orbital digraph* (TOD) or a $\mathbf{\Gamma}$-*orbital digraph*.

If G is self-paired then G is a *two-fold orbital graph* (TOG) or a $\mathbf{\Gamma}$-*orbital graph*.



This graph, has arc set $\mathbf{\Gamma}(1,2)$ *where* $\mathbf{\Gamma} = D_4 \times S_4 \leq S_4 \times S_4$.

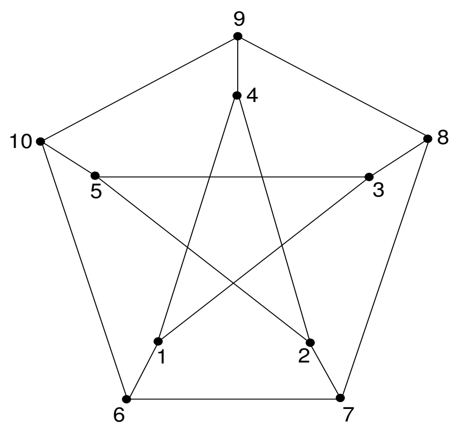
Although $\mathbf{\Gamma}$ *is not self-paired,* $A(G) = \mathbf{\Gamma}(1,2)$ *is.*

TF-ISOMOPRHISMS

Let G_1 and G_2 be two digraphs. Then (α, β) is a *two-fold isomorphism* from G_1 to G_2, (*TF-isomorphism*) $\iff$ α and β are bijections from $V(G_1)$ to $V(G_2)$ and $u \longrightarrow_{G_1} v \iff \alpha(u) \longrightarrow_{G_2} \beta(v)$ for every pair of vertices $u, v \in V(G_1)$.

If there is a TF-isomorphism between digraphs G_1 and G_2, we say that G_1 and G_2 are *TF-isomorphic* ($G_1 \cong^{TF} G_2$). A TF-isomorphism $(\alpha, \beta) : G_1 \to G_1$, that is, between a digraph and itself, is called a *TF-automorphism*.

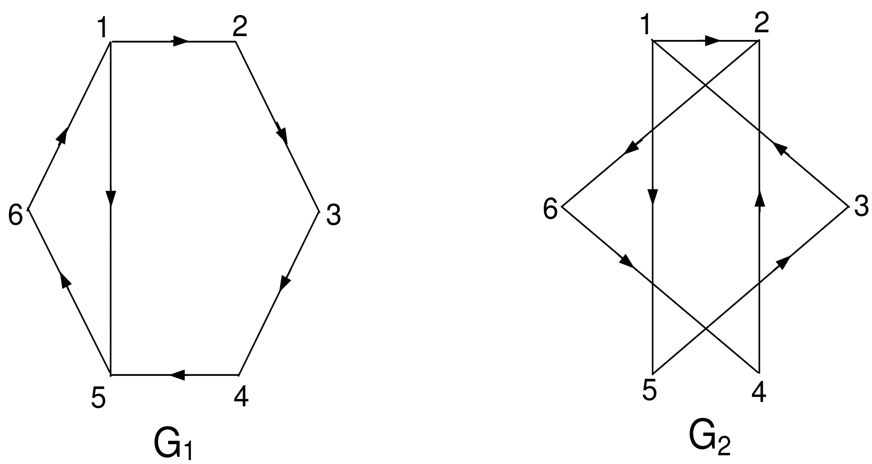
The set of TF-*automorphisms* of a digraph G is a group and it is denoted by $Aut^{TF}G$.



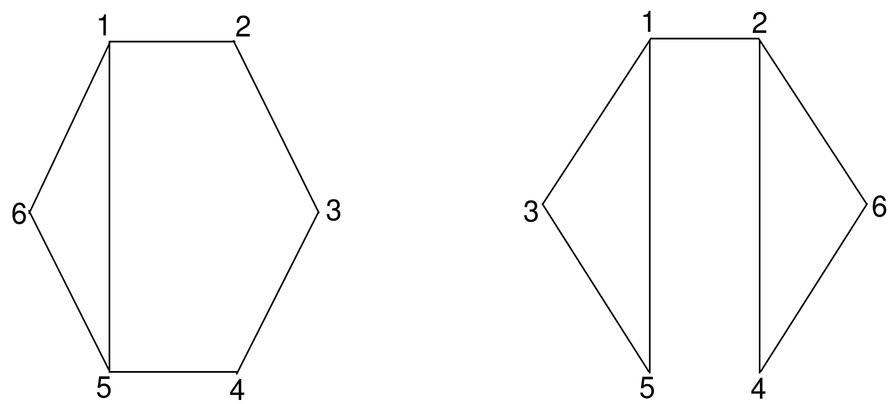
(α, β) *is a TF-automorphism of the Petersen graph where* α, β *are defined by*

$\alpha = \beta = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)$

(a)



(b) [underlying graphs :]



$\alpha = (1\ 2\ 3)(4)(5)(6)$, $\beta = (1\ 4)(2\ 5)(3\ 6)$ & (α, β) *is a TF-isomorphism from G_1 to G_2. But there is no TF-isomorphism between the corresponding underlying graphs.*

CANONICAL DOUBLE COVERINGS

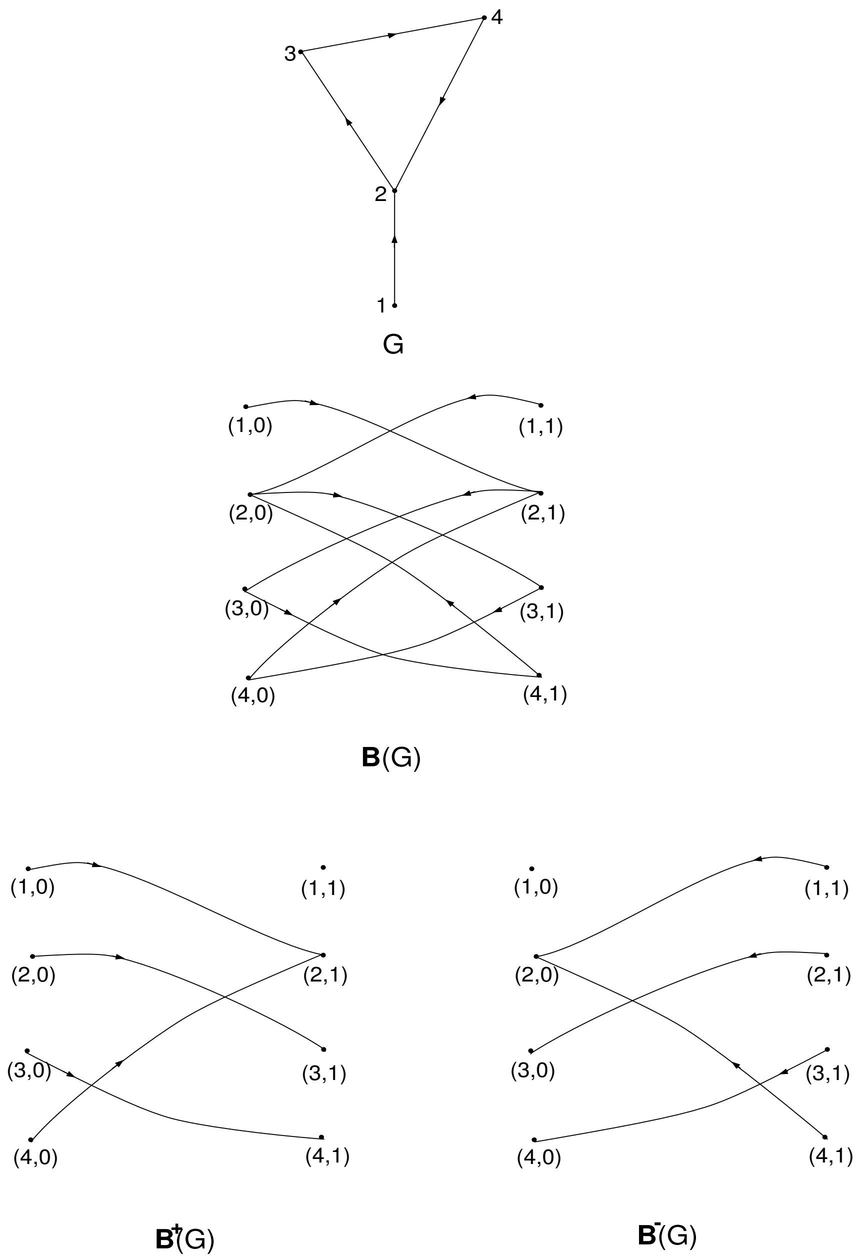
Let G be a digraph. The *canonical double covering* (CDC) of G is the digraph $\mathbf{B}(G)$, whose vertex set is $V(G) \times \mathbb{Z}_2$ such that there exists an arc

$$(u, \varepsilon) \longrightarrow (v, \varepsilon + 1) \iff u \longrightarrow_G v \text{ exists}$$
$$(\varepsilon, \in \mathbb{Z}_2).$$

Clearly $\mathbf{B}(G) = G \times K_2$, the categorical product of G and K_2.

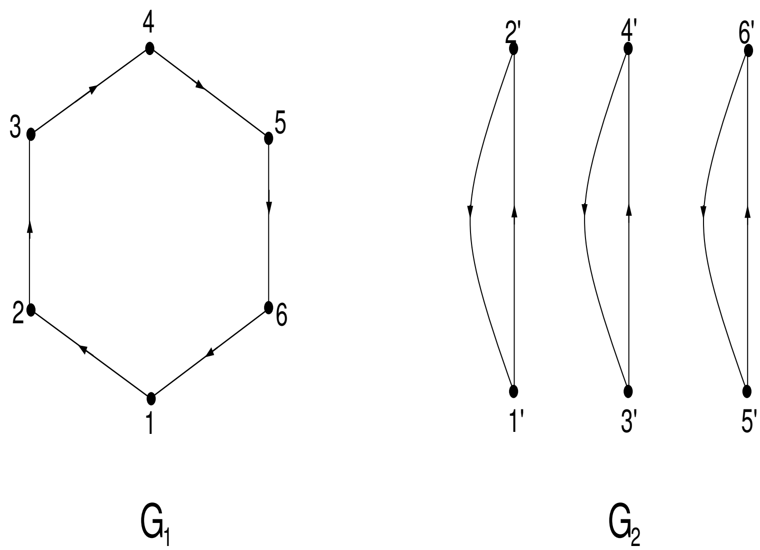
$\mathbf{B}(G)$ is bipartite.

If G is bipartite then $\mathbf{B}(G)$ is disconnected.



Digraphs G, $\mathrm{B}(G)$, $\mathrm{B}^{+}(G)$ *and* $\mathrm{B}^{-}(G)$.

Theorem. *Two digraphs* G_1 *and* G_2 *such that* $\mathbf{B}(G_1) \cong \mathbf{B}(G_2)$ *are TF-isomorphic. The converse holds in the case of graphs.*



$\mathbf{B}(G_1) \ncong \mathbf{B}(G_2)$. *However, there exists a TF-isomorphism* (α, β) *from* G_1 *to* G_2 *defined by* $\alpha = id$ *and* $\beta =$ (1 5 3)(2)(4)(6).

DISCONNECTED TWO-FOLD ORBITAL GRAPHS

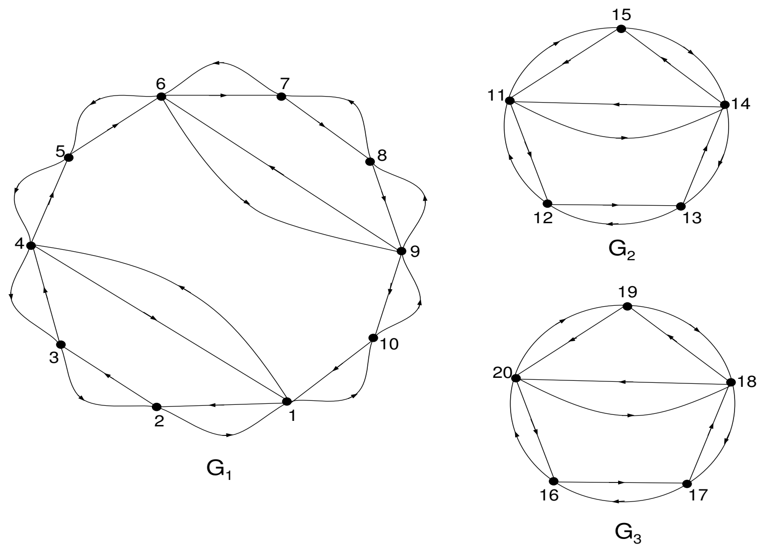
Theorem. *Let G be a TOG with no isolated vertices and let its connected components be $G_1, ..., G_k$ and:*

$$|V(G_1)| \geq |V(G_2)| \geq, ... \geq |V(G_k)|.$$

Then each $G_i (i = 1, ..., k)$ is still a TOG. Moreover :

(i) *if $|V(G_1) = |V(G_k)|$, then $G_1, G_2, ..., G_k$ are pairwise TF*-isomorphic *:*

(ii) *otherwise, there exists a unique index $r \in \{1, ...k - 1\}$ such that $G_1 \cong G_2 \cong ... \cong G_r \ncong^{TF} G_{r+1} \cong^{TF} ... \cong^{TF} G_k$ and $G_1 \cong \mathbf{B}(G_k)$*



Graph G has components G_1, G_2 and G_3.

G is a TOG where $\alpha =$

(1 11 20 6 9 14 18 4)(2 3 13 17 8 7 12 16)(5 15 19 10)

and $\beta =$

(1 4 14 18 9 6 11 20)(2 12 16 7 8 13 17 3)(5 10 15 19).

Theorem. *Let G be a disconnected TOG, with no isolated vertices and let its connected components be $G_1, ..., G_k$. If one of its components is bipartite then*

(i) *either all the components are isomorphic,*

(ii) *there exists a unique index $r \in \{1, ...k-1\}$ such that $G_1 \cong G_2 \cong ... \cong G_r \not\cong^{TF} G_{r+1} \cong^{TF} ... \cong^{TF} G_k$ and $G_1 \cong \mathbf{B}(G_k)$ where all G_i where $1 \le i \le r$ are bipartite and no G_j where $r+1 \le j \le k$ is bipartite.*

Corollary *A disconnected bipartite graph G is a TOG only if its components are all isomorphic.*

Disconnected TOG's and the cycle structures of α and β

Theorem. *Let G be a disconnected graph and let $|V(G)| = n$. Suppose that (α, β) is a TF-automorphism on G. Let $\alpha = \alpha_1\alpha_2 \ldots \alpha_r$ where α_i ($1 \leq i \leq r$) represent disjoint cycles of length $n_{\alpha i}$. Similarly let $\beta = \beta_1\beta_2 \ldots \beta_s$ where β_j ($1 \leq j \leq s$) represent a cycles of length $n_{\beta j}$. Then, if any two numbers from the set $\{n_{\alpha i}, n_{\beta j} | 1 \leq i \leq r, \quad 1 \leq j \leq s\}$ are relatively prime, the graph cannot be a TOG.*

AVOIDING LOOPS: χ-ORBITAL DIGRAPHS

χ-orbital digraphs

Let $\Gamma \leq S_n$ and $\chi : \Gamma \to \Gamma$ a homomorphism.

Then $D_\chi(\Gamma) = \{(\alpha, \chi(\alpha)) | \alpha \in \Gamma\} \leq \mathcal{S}$.

A *$D_\chi(\Gamma)-$two-fold orbital digraph* will be referred to as a χ-*orbital digraph* or a χ-TOD for short.

Let Γ be transitive and $\chi \in Aut(\Gamma)$. Define

$$M_\chi(\Gamma) = \{(\chi(\alpha))^{-1}\alpha \,|\, \alpha \in \Gamma\}$$

(Note: if Γ is abelian, then $M_\chi(\Gamma)$ is a subgroup of Γ.)

Theorem. *Let* $\Gamma \leq S_n$ *and* $\chi \in Aut\Gamma$.

(i) if $M_\chi(\Gamma) = \Gamma$, *then the graph* $\mathcal{D}_\chi(\Gamma)(u,v)$ *always has loops ;*

(ii) if $M_\chi(\Gamma) \neq \Gamma$ *and* Γ *is regular, then there exists* (u,v) *such that* $D_\chi(\Gamma)(u,v)$ *has no loops.*

Two elementary results about $M_\chi(\Gamma)$.

- If $|\Gamma|$ is odd, then $M_\chi(\Gamma) = \Gamma$, otherwise $M_\chi(\Gamma) < \Gamma$.

- If Γ is a nilpotent group of class 2 and χ is an inner automorphism of Γ, $\chi : \alpha \mapsto \gamma^{-1}\alpha\gamma$ then $M_\chi(\Gamma) < \Gamma$.

- **Problem 1.** Characterise those digraphs that are $\boldsymbol{\Gamma}$-orbital for a suitable group $\boldsymbol{\Gamma}$

- **Problem 2** Study the behaviour of the construction of TODs with respect to various graph-theoretical properties (here: disconnectedness).

- **Problem 3** Find ways in which the properties of a TOD can be controlled by making β depend on α.

- **Problem 4** Can $Aut^{TF}(G)$ give useful information about G?

- **Conjecture 1**. A disconnected TOG with no isolated vertices is a disconnected line graph of some graph G.

- **Conjecture 2**. Any disconnected TOG whose components are all isomorphic is an iterated line graph $L^n(G)$ of some graph G, where G is a disconnected bipartite TOG.

- **Conjecture 3**. The subgraph of a disconnected TOG made up of all isomorphic components of the disconnected TOG is an iterated line graph $L^n(G)$ of some graph G, where G is a disconnected bipartite TOG.

Conjectures due to Russell Mizzi.

http://staff.um.edu.mt/jlau/research