On a formula for the number of Euler trails for a class of digraphs

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Abstract

In this note we give an elementary combinatorial proof of a formula of Macris and Pulé for the number of Euler trails in a digraph all of whose vertices have in-degree and out-degree equal to 2.

1 Introduction

Let G be a connected *2-in-2-out digraph*, that is, a connected digraph in which each vertex has in-degree and out-degree both equal to 2 (loops and multiple arcs are allowed). Such a digraph is Eulerian. Let $Eu(G)$ be the set of Euler trails in G and let $\gamma \in Eu(G)$. Let $|V(G)| = n$. The $n \times n$ *intersection matrix* I_γ is constructed as follows. On the circumference of a circle mark $2n$ points. Traverse the trail γ starting at any vertex, which is given the label 1, and this way label all the vertices of G with the integers $1, 2, \ldots, n$ in the order in which they first appear along γ. Correspondingly, label the points on the circle with the integers $1, 2, \ldots, n$ in the same order as they appear along γ; each label therefore appears exactly twice on the circle. Join each pair of points labelled i by a chord (which is called the chord i). Suppose $j > i$. If chords i and j intersect, then $(I_\gamma)_{ij} = 1$ otherwise $(I_\gamma)_{ij} = 0$. The other elements of I_γ are defined by $(I_\gamma)_{ji} = -(I_\gamma)_{ij}$ and $(I_\gamma)_{ii} = 0$, making I_γ a skew-symmetric matrix.

In [3] the following theorem is proved.

Theorem 1.1 *The number of Euler trails in G is given by*

$$|Eu(G)| = \det(I + I_\gamma).$$

In this note we shall give an elementary combinatorial proof of this result.

Let X be an $n \times n$ matrix and suppose its rows and columns are indexed by $1, 2, \ldots, n$. The set $\{1, 2, \ldots, n\}$ will be denoted by N. If $S, T \subseteq N$, then $X[S; T]$ will denote the submatrix of X generated by the rows and columns indexed by S and T respectively. For brevity we sometimes write $X[i_1, \ldots, i_s; j_1, \ldots, j_t]$ instead of $X[\{i_1, \ldots, i_s\}; \{j_1, \ldots, j_t\}]$. When $S = T$ we write $X[S]$ for $X[S; S]$; therefore $X = X[N]$. The determinant $\det(X[\emptyset])$ is taken to be 1. The $n \times n$ identity matrix is denoted by I_n or simply I.

We shall need the following identity:

$$\det(I + X) = \sum_{\emptyset \subseteq S \subseteq N} \det(X[S]) \tag{1}$$

2 Decomposition of cycles by transpositions

A matrix very similar to I_γ as defined above has already been considered by others [2, 1]. Let τ be the cyclic permutation $(1 \quad 2 \quad \ldots \quad 2n)$ and let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be disjoint transpositions on the set $\{1, 2, \ldots, 2n\}$. Mark $2n$ points on the circumference of a circle labelled $1, 2, \ldots, 2n$ in cyclic order, and, for $1 \leq i \leq t$ join by the chord i the two elements transposed by σ_i. The $t \times t$ *link matrix* L is defined as follows: if chords i and j intersect then $L_{ij} = L_{ji} = 1$, otherwise $L_{ij} = L_{ji} = 0$; $L_{ii} = 0$.

For our purposes, the main result from [2, 1] is the following.

Theorem 2.1 *The permutation*

$$\sigma_1 \sigma_2 \ldots \sigma_t \tau$$

is a cycle iff

$$\det(L) = 1 \bmod 2.$$

Cohn and Lempel [2] proved this result when the transpositions are disjoint, and Beck [2] generalised it for arbitrary transpositions. The case of disjoint transpositions will be sufficient for our purposes.

The connection between this result and Euler trails in 2-in-2-out digraphs is as follows. Suppose $\sigma_1, \ldots, \sigma_t$ are disjoint and $t = n$. Consider the circle with the points labelled $1, 2, \ldots, 2n$ in that order. Relabel the points so that the two which are transposed by σ_i are now labelled i. The sequence of labels round the circle now gives an Euler trail γ in a 2-in-2-out digraph G. Moreover, we can assume that the transpositions $\sigma_1, \sigma_2, \ldots, \sigma_n$ are (possibly) renamed so that, starting from one of the points labelled 1, if $i < j$, then going round the circle a point labelled i is first encountered before one labelled j. In this case the matrix L is the same as the matrix I_γ except that every occurrence of a -1 in I_γ is replaced by a $+1$.

The property of whether or not $\sigma_1\sigma_2\ldots\sigma_t\tau$ is a cycle can also be interpreted as follows in terms of the Euler trail γ. Suppose the vertex i has the arcs a, b incident to it and the arcs c, d incident from it. Then any Euler trail takes one of two possibile routes at i: either arc a is immediately followed by arc c and arc b is immediately followed by arc d, or a is followed by d and b is followed by c. Now, given the Euler trail γ, one can ask whether or not there is another Euler trail γ^* in G such that, at every vertex, γ^* takes the alternative route to that taken by γ. Consider a slightly more general situation. Suppose $S \subset N = \{1, 2, \ldots, n\}$. Then we can ask whether or not there is an Euler trail γ_S which takes the alternative route to γ at each of the vertices in S and the same route as γ at each of the vertices in $N-S$. Clearly, if such a γ_S exists then it is unique. That is, if $C(S)$ denotes the set of Euler trails which take the alternative route to γ at each of the vertices in S and the same route as γ at each of the vertices in $N - S$, then $|C(S)| \in \{0, 1\}$.

Now, it can easily be seen that, for $S = \{i_1, i_2, \ldots, i_s\}$, γ_S exists iff $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_s}\tau$ is a cycle. This, and the fact that, if $\det(I_\gamma[S])$ equals 0 or 1, then $\det(I_\gamma[S]) = \det(L[S]) \bmod 2$, immediately gives the following corollary to Theorem 2.1.

Corollary 2.2 *Let* $\det(I_\gamma[S]) \in \{0, 1\}$. *Then* γ_S *exists iff* $\det(I_\gamma[S]) = 1$. *That is,* $\det(I_\gamma[S]) = |C(S)|$.

3 Proof of Theorem 1.1

The proof of Theorem 1.1 will follow as a result of the following.

Theorem 3.1 *For any* $S \subseteq N$,

$$\det(I_\gamma[S]) \in \{0, 1\}.$$

For, suppose Theorem 3.1 has been proved. Now,

$$\begin{aligned}
|Eu(D)| &= |\cup_{\emptyset\subseteq S\subseteq N} C(S)| \\
&= \sum_{\emptyset\subseteq S\subseteq N} |C(S)| \\
&= \sum_{\emptyset\subseteq S\subseteq N} \det(I_\gamma[S])
\end{aligned}$$

by Corollary 2.2. But this equals $\det(I + I_\gamma)$, by Equation 1.

We are therefore left with proving Theorem 3.1. Note that it is sufficient to prove the theorem for I_γ, since any submatrix $I_\gamma[S]$ is an intersection matrix in its own right, and therefore the result would also hold for it. Also, since I_γ is skew-symmetric, its determinant is 0 when n is odd. We may therefore assume that n is even, say $n = 2k, k \geq 1$. Our proof will be by induction on k, the result being clearly true for $k = 1$. Our methods will be very similar to those employed in [2].

If no two chords intersect, then $\det(I_\gamma) = 0$. We may therefore assume that at least two chords intersect. Also, we may assume that the initial vertex in the labelling has been chosen such that chords 1 and 2 intersect. Taking the same point of view as in [2], these two chords correspond to the transpositions σ_1, σ_2 respectively. Since the chords corresponding to σ_1, σ_2 intersect, the permutation $\tau' = \sigma_1\sigma_2\tau$ is also a cycle.

Crucial to the inductive step is the consideration of how the linkage between the transpositions $\sigma_3, \ldots, \sigma_n$ with respect to τ' is related to their linkage with respect to τ. In terms of matrices, if K is the intersection matrix of $\sigma_3, \ldots, \sigma_n$ with respect to τ', we need to investigate how K is related to I_γ.

The arguments we use are very similar to those in [2], but here more care has to be taken because the relationship between the transpositions can change not only in terms of whether or not their chords remain linked/unlinked when going over from τ to τ', but the sign of the linkage can also change, as explained below.

The relationship between τ and τ' is shown schematically in Figure 1. The first circle in Figure 1 represents τ (in a clockwise sense), and the transpositions σ_1, σ_2 are represented by chords 1 and 2. The second circle represents τ'.

Figure 1

Now, it can very well happen that the order in which chords $3, 4, \ldots, n$ appear round the circle representing τ' need not be the same as that round the circle representing τ. Therefore we need here to consider more carefully the order in which chords are labelled when constructing the intersection matrix I_γ. In the definition given above, this order corresponds to the order in which

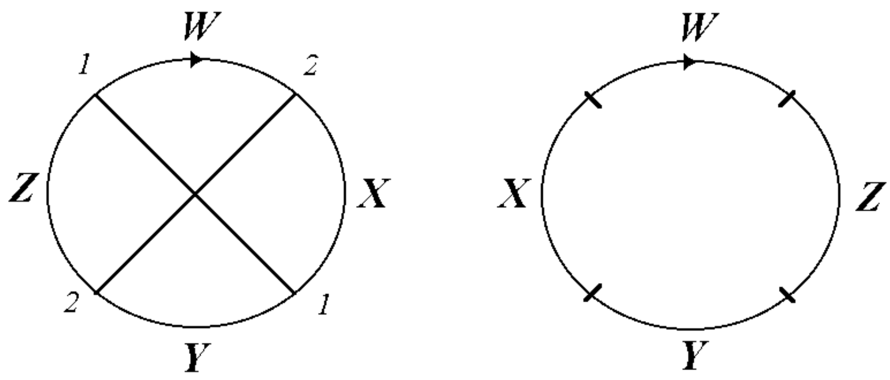

Figure 1: Representations of τ and $\sigma_1\sigma_2\tau$.

the ends of the chords first appear when going round the circle. Changing this order amounts to a simultaneous interchange of rows and columns of I_γ; I_γ would still be skew-symmetric and $\det(I + I_\gamma)$ would be unchanged. But the matrix I_γ would not necessarily have all its positive terms above the main diagonal and all the negative ones below—the definition of I_γ needs only to be slightly extended as follows: Suppose the chords are labelled $1, 2, \ldots, n$ in an arbitrary fashion, and consider the circle traversed starting from one end of chord 1 (we may suppose that the circle is traversed in a clockwise sense). Let $j > i$. If chords i and j do not intersect, then $(I_\gamma)_{ij} = 0$. If chords i and j do intersect, then $(I_\gamma)_{ij} = 1$ if an end of chord i first appears before an end of chord j as the circle is traversed and $(I_\gamma)_{ij} = -1$ otherwise. The other elements of I_γ are defined by $(I_\gamma)_{ji} = -(I_\gamma)_{ij}$ and $(I_\gamma)_{ii} = 0$.

Now let $A = I_\gamma[1, 2]$. Since chords 1 and 2 intersect,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore $A^2 = -I$. Let $B = I_\gamma[1, 2; 3, \ldots, n]$ and let $C = I_\gamma[3, \ldots, n]$. Therefore

$$I_\gamma = \left(\begin{array}{c|c} A & B \\ \hline -B^t & C \end{array} \right).$$

Let K be the intersection matrix of the chords $3, \ldots, n$ with respect to τ'. The rows and columns of K correspond to the chords in the order $3, \ldots, n$, which is not necessarily the order in which the chords appear along the circle representing τ'; therefore, as discussed above, the matrix K could have negative terms above and below the main diagonal. We can now proceed in a similar vein to [2], with due care given to signs.

Let $j > i$. A consideration of the possible intersections which can arise between chord i and chord j with respect to τ will verify that

$$K[i,j] = B[i,j]^t A^{-1} B[i,j] \pm C[i,j] \tag{2}$$

The term $C[i,j]$ must appear with a minus sign in the above formula if it happens that (with respect to γ) chords i and j intersect and exactly one of them does not intersect with either of chords 1 or 2. (Referring to Figure 1, this means that both ends of the chord are in the same sector X, Y or Z.) Let us call *special* a chord which does not intersect either one of chords 1 and 2 with respect to γ. Note that chord k is special iff $B_{1,k} = B_{2,k} = 0$.

Let J be the $(n-2) \times (n-2)$ diagonal matrix whose rows and columns are indexed by $3, 4, \ldots, n$ and such that $J_{kk} = -1$ if chord k is special and $J_{kk} = 1$ if chord k is not special. From the above equation we deduce that

$$K = B^t A^{-1} B + JCJ \tag{3}$$

Note that if chords i and j do not intersect with respect to γ then $C[i,j]$ is the zero matrix, so that it is immaterial whether there is a plus or a minus in Equation 2. Also, if both chords i and j are special and they do intersect with respect to γ (so that there should be a plus in Equation 2), then $(JCJ)[i,j] = C[i,j]$, as required. Therefore Equation 3 does in fact follow from Equation 2. Note also that, since $B_{1,k} = B_{2,k} = 0$ when chord k is special, $BJ = B$.

Now let

$$R = \left(\begin{array}{c|c} I_2 & AB \\ \hline 0 & J \end{array} \right).$$

Then, using the facts that $BJ = B$ and $A^2 = -I$ we obtain

$$R^t I_\gamma R = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B^t A^{-1} B + JCJ \end{array} \right) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & K \end{array} \right).$$

Therefore $\det(I_\gamma) = \det(K)$. But K is an intersection matrix, and so the result follows by induction.

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