Constructing graphs with several pseudosimilar vertices or edges

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Abstract

Some of the most interesting problems connected with pseudosimilarity in graphs concern the construction of graphs with large sets of pseudosimilar vertices or edges. This can be understood in two ways: Either the graph contains a large set of vertices or edges which are mutually pseudosimilar or else for every vertex (edge) in the graph there is another vertex (edge) to which it is pseudosimilar. We shall survey the methods used to construct such graphs and on the way we shall also discuss some related results and point out some unanswered questions.

1 Introduction

All graphs considered will be finite, simple and undirected, unless otherwise stated. The vertex-set and the edge-set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If v is a vertex in G, then $G - v$ denotes the subgraph of G obtained by removing v and all edges incident to v; if e is an edge of G then $G - e$ denotes the subgraph obtained by removing the edge e.

Two vertices u, v in a graph G are said to be *similar* if there is an automorphism of G which maps u into v. The vertices u, v are said to be *removal-similar* if the subgraphs $G - u$ and $G - v$ are isomorphic. If u and v are removal-similar but not similar, then they are called *pseudosimilar*. In this case we sometimes say that v is a *pseudosimilar mate* of u and vice-versa. If $S \subset V(G)$ such that any two vertices in S are pseudosimilar mates, then we say that the vertices of S are *mutually pseudosimilar* in G.

Pseudosimilar edges are similarly defined, as are the terms pseudosimilar mates for pairs of edges and mutually pseudosimilar for sets of edges.

The reason why pairs of pseudosimilar vertices arise is quite well understood in terms of a sort of truncation of cyclic symmetry. Thus, take a graph H with vertices u and v and an automorphism α of H such that $\alpha^t(u) = v$ for some $t > 1$ and $\alpha^r(u) \neq v$ for $1 \leq r < t$. Then u and v are removal-similar in $G = H - \{\alpha(u), \ldots, \alpha^{t-1}(u)\}$; if moreover they also happen to be not similar, then we have a pair of pseudosimilar vertices. Godsil and Kocay [7] showed that, in fact, every pair of pseudosimilar vertices can be obtained this way.

Theorem 1 ([7]) *Let u and v be two pseudosimilar vertices in a graph G. Then G is an induced subgraph of some graph H such that H has an automorphism α with $\alpha(G-v) = G-u$ and $\alpha^t(u) = v$, and such that $V(H)-V(G) = \{x_1, \ldots, x_r\}$, where $x_i = \alpha^{t+i}(u)$ and $\alpha(x_r) = u$.*

Therefore the most interesting questions and constructions involve graphs with several pseudosimilar vertices or edges. The two situations we shall be investigating are the construction of graphs in which every vertex (edge) has a pseudosimilar mate and graphs with large sets of mutually pseudosimilar vertices (edges).

A general survey about pseudosimilarity can be found in [18]. Some work presented here has been carried out since the publication of that survey. Graph theoretic terms used but not defined in this paper are standard and can be found in any graph theory text such as [9].

2 Every vertex can have a pseudosimilar mate: The KSS construction.

The question of whether or not in a graph every vertex can have a pseudosimilar mate has been settled since 1981 [12]. The solution to this question turns out to be a simple corollary of the solution to another problem about symmetries of graphs, namely the construction of *graphical regular representations (GRR)* of groups of odd order. A graph G is said to be a GRR of a group Γ if $\mathrm{Aut}(G) \simeq \Gamma$ and $\mathrm{Aut}(G)$ acts regularly on $V(G)$, that is, it is transitive on $V(G)$ and the stabiliser of any $v \in V(G)$ is trivial. Except for a finite number of known groups, all finite, nonabelian groups which are not generalised dicyclic groups have GRRs. A number of authors contributed towards obtaining this result, but here we shall only be requiring GRRs for groups of odd order (it follows, see [2] for example, that such groups must be nonabelian).

Theorem 2 ([10]) *Except for one group of order 27, all nonabelian groups of odd order have GRRs.*

Using the existence of GRRs for groups of odd order enabled Kimble, Schwenk and Stockmeyer to construct graphs in which every vertex has a pseudosimilar mate.

Theorem 3 ([12]) *There are infinitely many graphs in which every vertex has a pseudosimilar mate.*

Proof Let Γ be a group of odd order and let H be a GRR of Γ. We note that, since the stabiliser of any vertex of H under the action $\mathrm{Aut}(H) \simeq \Gamma$ is the identity element of Γ, it follows that if r is any vertex of H, then $G = H - r$ has the identity automorphism group.

Now, let v be any vertex in G. There is an automorphism α of H mapping r to v. The vertices $\alpha^{-1}(r)$ and $v = \alpha(r)$ are distinct, because otherwise α

would contain a cycle of length 2, which is impossible since Γ has odd order. Since α^{-1} maps $\{v, r\}$ onto $\{r, \alpha^{-1}(r)\}$, it follows that $G - v = H - r - v \simeq H - \alpha^{-1}(r) - r = G - \alpha^{-1}(r)$; that is, $v = \alpha(r)$ and $\alpha^{-1}(r)$ are removal-similar in G. But G has the identity automorphism group, therefore v and $\alpha^{-1}(r)$ are pseudosimilar. $\square$

We shall refer to this construction as the *KSS construction.* In [12], Kimble, Schwenk and Stockmeyer also gave some nice examples illustrating the use of the above theorem.

One question which the above result brings up is whether or not the KSS construction is the only one which gives graphs all of whose vertices have pseudosimilar mates.

Question 1 *Is there a characterisation analagous to Theorem 1 of graphs all of whose vertices have a pseudosimilar mate? If in a graph G all vertices have a pseudosimilar mate, is it always possible to extend G to a vertex-transitive graph by adding only one new vertex? In particular, are all such graphs obtainable via the KSS construction?*

3 Every edge can have a pseudosimilar mate: Adapting the KSS construction.

Finding graphs in which every edge has a pseudosimilar mate proved to be more elusive. First attempts [11, 18] only managed to show that there are families of graphs of order n such that, as n increases, the proportion of edges in the graph having a pseudosimilar mate tends to 1. However, in 1996, using graphs constructed by Alspach and Xu [1], Lauri and Scapellato [21] proved the following.

Theorem 4 ([21]) *There are infintely many graphs in which every edge has a pseudosimilar mate.*

The idea is to adapt the KSS construction as follows. Let H be a graph with an odd number of edges and whose automorphism group acts regularly on its edge-set. Then, as in Theorem 3, deleting from H any edge gives a graph all of whose edges have a pseudosimilar mate.

The problem is to find such graphs H. Families of graphs with these properties were, in fact, constructed in [1] and a special case of this family can be described as Cayley graphs in the following way. (We recall that, if Γ is a group and $S \subset \Gamma$ with $S^{-1} = S$, $1 \notin S$ and $\Gamma = \langle S \rangle$, then the *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ is the graph with vertex-set equal to Γ and in which two vertices x, y are adjacent if and only if $y = xs$ for some $s \in S$.)

Let p be a prime number with $p \equiv 1 \mod 3$ and $p \equiv 1 \mod 5$. Let Γ_{3p} be the group defined by

$$\Gamma_{3p} = \langle b, c | b^3 = c^p = 1, c^b = b^{-1}cb = c^r \rangle$$

where r is such that $r^3 = 1 \mod 3$. Let t be such that $t^5 = 1 \mod p$, and let a be the automorphism of Γ_{3p} defined by $b^a = b$ and $c^a = c^t$. Let

$$T = \{c^a, c^{a^2}, c^{a^3}, c^{a^4}, c^{a^5} = c\}$$

and

$$S = bT \cup T^{-1}b^{-1} = bT \cup T^{-1}b^2.$$

Let H_{3p} be the Cayley graph $\mathrm{Cay}(\Gamma_{3p}, S)$.

Theorem 5 ([1]) *The automorphism group of the Cayley graph H_{3p} constructed above acts regularly on its edge-set.*

This Cayley graph has order $3p$ and degree 10, therefore it has an odd number, $15p$, of edges, as required. Also, by Dirichlet's Theorem on primes in an arithmetic progression (see [4], for example), there is an infinite number of primes in the arithmetic progression $\{1 + 15k : k = 0, 1, 2, \dots\}$ and therefore an infinite number of Cayley graphs H_{3p} can be constructed as above. This therefore proves Theorem 4.

The smallest value of p for which the above construction works is $p = 31$ giving a Cayley graph with 465 edges and therefore a graph with 464 edges, all of them paired by pseudosimilarity.

In the above construction, the Cayley graphs H_{3p} are all $\frac{1}{2}$-transitive, that is, the automorphism group is transitive on the vertices and the edges, but not on the directed arcs. Since what we need is a graph whose automorphism group acts regularly on its edges, one question which arises following the previous construction is whether or not it is possible to obtain a graph which is not vertex-transitive but whose automorphism group has the required action on the edge-set—such a graph would, of course, have to be bipartite. A graph of this type was constructed in [19] and we shall now briefly describe it.

We first give a few general definitions and results. The motivating idea behind these is the well-known characterisation, due to Sabidussi [23], of vertex-transitive graphs in terms of coset graphs.

Let Γ be a group and $\mathcal{H}, \mathcal{K}$ two subgroups of Γ. Let S be a subset of Γ. Define the graph $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K}, S)$ as follows: Its vertices are the left cosets of $\mathcal{H}$ and of $\mathcal{K}$; two cosets $x\mathcal{H}$ and $y\mathcal{H}$ are adjacent if and only if $y^{-1}x \in \mathcal{K}S\mathcal{H}$. If, moreover, $S \subseteq \mathcal{K}\mathcal{H}$, that is, $\mathcal{K}S\mathcal{H} = \mathcal{K}\mathcal{H}$, then we denote $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K}, S)$ simply by $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$.

If $\mathcal{H} \cap \mathcal{K} = \{1\}$, then any two cosets $x\mathcal{H}$, $y\mathcal{K}$ are either disjoint or have exactly one element in common. In this case, $x\mathcal{H}$ and $y\mathcal{K}$ are adjacent in $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ if and only if they are not disjoint, that is, all edges of $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ are of the form $\{t\mathcal{H}, t\mathcal{K}\}$, where t is the element common to both cosets. Another useful way to look at adjacencies in $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ when $\mathcal{H} \cap \mathcal{K}$ is trivial is as follows: The coset $x\mathcal{H}$ is adjacent to all the cosets $xh\mathcal{K}$, for all $h \in \mathcal{H}$ (all these cosets are distinct); similarly, the coset $y\mathcal{K}$ is adjacent to all the cosets $yk\mathcal{H}$ for all

$k \in \mathcal{K}$. Clearly, the degrees of the cosets $x\mathcal{H}$ and $y\mathcal{K}$ as vertices in $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ are $|\mathcal{H}|$ and $|\mathcal{K}|$, respectively.

The following two results are not difficult to prove.

Theorem 6 *Let G be a graph whose vertex-set is partitioned into two orbits V_1, V_2 under the action of the automorphism group Γ. Let $\mathcal{H}$ be the stabiliser of the vertex $u \in V_1$ and $\mathcal{K}$ the stabiliser of the vertex $v \in V_2$. Let S be the set of all those permutations $\alpha \in \Gamma$ such that $\alpha(u)$ is adjacent to v. Then G is isomorphic to $Cos(\Gamma, \mathcal{H}, \mathcal{K}, S)$. Moreover, if G is edge-transitive then $S \subseteq \mathcal{K}\mathcal{H}$, that is, G is isomorphic to $Cos(\Gamma, \mathcal{H}, \mathcal{K})$.*

Theorem 7 *Let $G = Cos(\Gamma, \mathcal{H}, \mathcal{K})$. For $t \in \Gamma$, let λ_t denote the action of left translation by t on the left cosets of $\mathcal{H}$ and $\mathcal{K}$. Then λ_t is an automorphism of G; this action is transitive on the edges of G. Suppose ϕ is an automorphism of Γ which fixes setwise both $\mathcal{H}$ and $\mathcal{K}$. Let $\hat{\phi}$ denote the induced action on the cosets of $\mathcal{H}$ and $\mathcal{K}$. Then $\hat{\phi}$ is an automorphism of the graph G.*

From these two theorems it is clear that to obtain a graph whose automorphism group acts regularly on the edges but non-transitively on the vertices we need to find a coset graph $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ such that no automorphism of the group fixes $\mathcal{H}$ and $\mathcal{K}$. Of course we also require the graph to be connected, therefore $\mathcal{H} \cup \mathcal{K}$ must generate all of Γ. We can now describe the graph constructed in [19].

Let Ξ be the group of order $3 \cdot 5 \cdot 31$ defined as follows

$$\Xi = \langle a, w, c | a^5 = w^3 = c^{31} = 1, wa = awc, ca = ac^2, cw = wc^{25} \rangle.$$

Now let $\mathcal{H}$ be the cyclic subgroup generated by a and let $\mathcal{K}$ be the cyclic subgroup generated by w. Let $H = \mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$. This graph is edge-transitive but not vertex-transitive since the cosets of $\mathcal{H}$ have degree 5 whereas the cosets of $\mathcal{K}$ have degree 3. Moreover, it is not difficult to check that no nontrivial automorphism of the group Γ fixes $\mathcal{H} \cup \mathcal{K}$, therefore there is reason to hope that, in fact, the full automorphism group of H is Ξ, that is, the automorphism group of H acts regularly on the edges. For this it is required to show that the stabiliser of any edge is trivial.

It turns out that the girth of H is 8 and that there are exactly eight cycles of length 8 containing any edge. A detailed consideration of these possible cycles leads to Figure 1, which shows all the 8-cycles passing through any of the three edges incident to $\mathcal{K}$ (and also the names of some of the vertices). By a more detailed consideration of the configuration shown in Figure 1 and using the fact that H is edge-transitive it is shown in [19] that if an automorphism of H fixes the edge $\{\mathcal{H}, \mathcal{K}\}$ then it must be trivial, as required.

The graph H in this last construction has 248 vertices and 465 edges, and therefore this again gives a graph with 464 edges all of which are paired by pseudosimilarity. The following question therefore naturally arises.

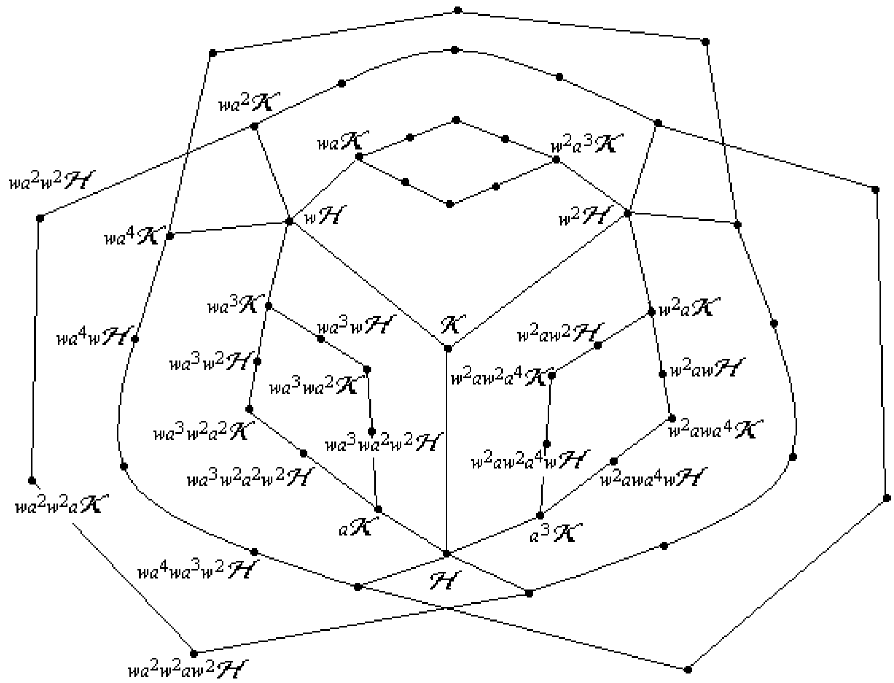

Figure 1: All the 8-cycles passing through the edges incident to $\mathcal{K}$

Question 2 *Are there graphs with less than 464 edges in which every edge has a pseudosimilar mate?*

Also, the non-vertex-transitive graph whose automorphism group acts regularly on its edges, and which was used in the previous construction, could very well be the first such graph in an infinite family, analogous to $H_{3p}, p = 31$, for Theorem 5. Therefore one can ask,

Question 3 *Find an infinite family of graphs which are not vertex-transitive but whose automorphism groups act regularly on the respective edge-sets.*

Finally, one can ask for pseudosimilar edges a question analogous to Question 1 of the previous section.

Question 4 *Is there a characterisation, analogous to Theorem 1, of graphs all of whose vertices have a pseudosimilar mate? Are all such graphs obtainable via the KSS construction adapted for edges?*

4 Cayley line-graphs

Whereas the problem of constructing graphs in which every vertex has a pseudosimilar mate turned out to be an easy application of the existence of GRRs for finite groups of odd order, the analogous problem for pseudosimilar edges was more difficult because there it was not sufficient to know that a group had a GRR; the GRR had to have some particular structure—not any GRR of the group would do.

The situation is perhaps best understood in terms of line-graphs, for now, since we are looking for a graph H whose automorphism group acts regularly on its edge-set, the line graph $L(H)$ is a GRR of its automorphism group (which is isomorphic to $\mathrm{Aut}(H)$). Therefore $L(H)$ is a Cayley graph of $\mathrm{Aut}(H)$ (see [2], for example). Hence we are now looking for particular Cayley graphs, namely those which are line-graphs.

It is therefore natural to ask, in this context, what form the set S must take for the Cayley graph $\mathrm{Cay}(\Gamma, S)$ to be a line graph. The answer is given by the next theorem.

Theorem 8 *Let Γ be a finite group and let $S \subseteq \Gamma$ with $S^{-1} = S$, $1 \notin S$ and $\Gamma = \langle S \rangle$. Let $S^* = S \cup \{1\}$. Then $G = Cay(\Gamma, S)$ is a line-graph if and only if $S^* = S_1 \cup S_2$ such that:*

1. *$S_1 \cap S_2 = \{1\}$,* **and**
2. **either** *both S_1 and S_2 are subgroups of Γ,*
3. **or else** *$S_1 = \mathcal{H} \cup \mathcal{H}a$ and $S_2 = a^{-1}\mathcal{H}a \cup a^{-1}\mathcal{H}$, for some $\mathcal{H} \leq \Gamma$ and $a \in \Gamma$ with $\mathcal{H} \cap a^{-1}\mathcal{H}a = \{1\}$.*

Proof We first recall the characterisation of line-graphs in terms of the Krausz decomposition of its edge-set (see [9]), namely, that a 2-connected graph (as is our Cayley graph G since it is vertex-transitive) is a line-graph if and only if its edges can be partitioned so that the edges in each part induce a complete graph and every vertex is incident to edges from exactly two parts of the partition. In the case of G (again since it is vertex-transitive) it is a line-graph if and only if the neighbours of one of its vertices v_0 together with v_0 induce two complete graphs which intersect only in v_0. We can take v_0 to be the vertex 1, whose set of neighbours is S. Therefore G is a line-graph if and only if $S^* = S_1 \cup S_2$ such that

(i) $S_1 \cap S_2 = \{1\}$, and

(ii) if $s_1, s_2 \in S^*$ then $s_1^{-1}s_2 \in S^*$ if and only if both s_1 and s_2 are in S_1 or S_2.

If Condition 1 and one of Conditions 2 or 3 of the theorem hold, then so do Conditions (i) and (ii), that is, G is a line-graph. Therefore, for the converse, suppose G is a line-graph, that is, Conditions (i) and (ii) hold. In the sequel, for $x, y \in S^*$ we shall use the notation $x \sim y$ to denote that x and y are both in S_1 or in S_2.

We now make two observations:

Observation 1.
Suppose S_i ($i = 1$ or 2) contains two subgroups $\mathcal{A}, \mathcal{B}$ of Γ. Then S_i also contains the subgroup $\mathcal{C} = \langle \mathcal{A} \cup \mathcal{B} \rangle$ generated by $\mathcal{A} \cup \mathcal{B}$. For, by (ii) and since each of $\mathcal{A}, \mathcal{B}$ contains the inverse of each of its elements, we have that for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the elements $ab = (a^{-1})^{-1}b$ and $ba = (b^{-1})^{-1}a$ are both in S^*. Moreover, since $a^{-1}(ab)$ and $b^{-1}(ba)$ are in S^* and $a, b \in S_i$, then, by (ii), ab and ba are also in S_i. Therefore, if an element like $w = a_1 b_2 a_3 \ldots a_{n-1} b_n$ is a product of elements $a_i \in \mathcal{A}, b_j \in \mathcal{B}$, we can show that $w \in S_i$ by induction on n: since a_1^{-1} and $b_2 a_3 \ldots a_{n-1} b_n$ are both in S_i then $w = (a_1^{-1})^{-1} b_2 a_3 \ldots a_{n-1} b_n$ is in S^*; also, since $a_1 \in S_i$ and $a_1^{-1} w$ is in S^*, then w is also in S_i, by (ii).

Therefore S_i contains a subgroup of Γ which is maximal in the sense that it contains every subgroup of Γ found in S_i. We denote this maximal subgroup by $\mathcal{H}_i$.

Observation 2.
Each S_i is the union of right cosets of $\mathcal{H}_i$. For, let $g \in S_i$. Then, for any $h \in \mathcal{H}_i$, $h^{-1}g \in S^*$, that is, $\mathcal{H}_i^{-1}g = \mathcal{H}_i g \subset S^*$. But, for any $h_1 \in H_i$ and $h_2 g \in H_i g$, we have that $h_1^{-1} h_2 g \in S^*$. Therefore $h_2 g$ must be in S_i, that is, $\mathcal{H}_i g \in S_i$. However, $g \in \mathcal{H}_i g$, that is, any element of S_i is in some right coset of $\mathcal{H}_i$.

We now claim that any two elements of S_i not contained in $\mathcal{H}_i$ must be in the same right coset of $\mathcal{H}_i$.

Consider, without loss of generality, S_1. Let $x, y \in S_1, x \neq y$ and let $x^{-1}y = z \in S^*$. From the relation $xz = y$ and $yz^{-1} = x$ it follows that $x^{-1} \sim z$ and $y^{-1} \sim z^{-1}$. There are now four cases to consider. (Note that below we use the fact that if both a and a^{-1} are in S_i then so is $\langle a \rangle$.)

Case I: $z \in S_1$.

Case I.1: $z^{-1} \in S_1$.

Therefore $x^{-1}, y^{-1} \in S_1$, and so, $\langle x \rangle$ and $\langle y \rangle$ are in S_1. Therefore all pairs of elements $x, y \in S_1$ such that $x^{-1}y = z \in S_1$ with z^{-1} also in S_1 must be in $\mathcal{H}_1$.

Case I.2: $z^{-1} \in S_2$.

Therefore $\langle x \rangle \subset S_1$ and so $x \in \mathcal{H}_1$. Moreover, $y^{-1} \notin S_1$, therefore $y \notin \mathcal{H}_1$, that is, y is in a nontrivial coset $\mathcal{H}_1 y$ of H_1 contained in S_1.

Case II: $z \in S_2$.

Case II.1: $z^{-1} \in S_1$.

Therefore $y, y^{-1} \in S_1$, that is, $\langle y \rangle \subset S_1$. Again, $y \in \mathcal{H}_1$ and x is in a nontrivial right coset $\mathcal{H}_1 x$ contained in S_1.

Case II.2: $z^{-1} \in S_2$.

Therefore $x^{-1}, y^{-1} \in S_2$. Consider $xy^{-1} = (x^{-1})^{-1} y^{-1} \in S$. But $x^{-1} \cdot xy^{-1} = y^{-1}$§. Therefore $x \sim xy^{-1}$, that is, $xy^{-1} \in S_1$. Similarly, yx^{-1} S_1.

Therefore S_1 contains $\langle xy^{-1}\rangle \leq \mathcal{H}_1$. Therefore x, y are in non-trivial cosets of $\mathcal{H}_1$ (nontrivial since $x^{-1}, y^{-1} \notin S_2$). But $\mathcal{H}_1 y$ contains $xy^{-1}\cdot y = x$, that is, x and y are in the same nontrivial right coset of $\mathcal{H}_1$.

This proves our claim, and hence we can say that $S_1 = \mathcal{H}_1$ or $S_1 = \mathcal{H}_1 \cup \mathcal{H}_1 a$ and similarly $S_2 = \mathcal{H}_2$ or $S_2 = \mathcal{H}_2 \cup \mathcal{H}_2 b$. If $S_1 = \mathcal{H}_1$ and $S_2 = \mathcal{H}_2$ then we are done. So, suppose $S_1 = \mathcal{H}_1 \cup \mathcal{H}_1 a$ with $a \notin \mathcal{H}_1$. Therefore $a^{-1} \notin S_1$, otherwise $\langle a\rangle \subset S_1$ and a would therefore be in $\mathcal{H}_1$.

Hence $a^{-1} \in S_2$, and since $a^{-1} \notin \mathcal{H}_2$ then $a^{-1} \in \mathcal{H}_2 b$, which is therefore $H_2 a^{-1}$. That is, $S_2 = \mathcal{H}_2 \cup \mathcal{H}_2 a^{-1}$.

Now, for all $g \in \mathcal{H}_1 a$, g^{-1} is in S_2 but not in $\mathcal{H}_2$ (since $g \notin S_2$). Therefore $g^{-1} \in \mathcal{H}_2 a^{-1}$, so that $(\mathcal{H}_1 a)^{-1} \subseteq \mathcal{H}_2 a^{-1}$. Similarly, $(\mathcal{H}_2 a^{-1})^{-1} \subseteq \mathcal{H}_1 a$. Therefore $(\mathcal{H}_1 a)^{-1} = a^{-1}\mathcal{H}_1 = \mathcal{H}_2 a^{-1}$, hence $H_2 = a^{-1}\mathcal{H}_1 a$. Therefore $S_1 = \mathcal{H}_1 \cup \mathcal{H}_1 a$ and $S_2 = a^{-1}\mathcal{H}_1 a \cup a^{-1}\mathcal{H}_1$, as required. □

(The line-graph of the Cayley graph H_{3p} for $p = 5$ considered in the previous section is, in fact, the Cayley graph $\mathrm{Cay}(\Xi, S)$ (where Ξ is the group considered later in the same section) with $S^* = \mathcal{H} \cup \mathcal{H}w \cup w^{-1}\mathcal{H}w \cup w^{-1}\mathcal{H}$, where $\mathcal{H} = \langle a\rangle$.)

The problem of finding a graph whose automorphism group acts regularly on its edges can therefore be regarded as a problem of finding a Cayley graph $\mathrm{Cay}(\Gamma, S)$ which is a GRR and such that S has the special form described in the previous theorem. From this theorem, the simplest way to guarantee that $\mathrm{Cay}(\Gamma, S)$ is a line-graph is to let $S = \mathcal{H} \cup \mathcal{K} - \{1\}$ where $\mathcal{H}, \mathcal{K}$ are subgroups of Γ with trivial intersection. (In this case, if $\mathrm{Cay}(\Gamma, S)$ is the line-graph $L(H)$ of H then H is the graph $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ as defined in the previous section.)

Now, for the Cayley graph to be a GRR it is necessary that no automorphism of Γ fixes S. This necessary condition is not, in general, sufficient. The following result of Godsil [6], however, affirms that for a wide class of p-groups this simple condition is also sufficient to guarantee that the Cayley graph is a GRR.

Theorem 9 ([6]) *Let Γ be a finite p-group which admits no homomorphism onto the wreath product of $\mathbb{Z}_p$ by $\mathbb{Z}_p$. Let $S \subset \Gamma$, $S = S^{-1}$ and $\Gamma = \langle S\rangle$ such that no nontrivial automorphism of Γ fixes S. Then the Cayley graph Cay(Γ, S) is a GRR of Γ.*

Godsil's theorem and the above discussion have led Lauri and Scapellato [21] to pose the following question:

Question 5 *Does there exist a p-group Γ (p an odd prime) having two subgroups $\mathcal{H}, \mathcal{K}$ with the following properties: (i) $\mathcal{H} \cap \mathcal{K} = \{1\}$, (ii) $\Gamma = \langle \mathcal{H} \cup \mathcal{K}\rangle$, and (iii) no nontrivial automorphism of Γ fixes $\mathcal{H} \cup \mathcal{K}$ setwise?*

If Γ is not a p-group then finding such subgroups is possible. For example, if Ξ is again the group defined in the previous section, then it is routine to check that the subgroups $\mathcal{H} = \langle a\rangle$ and $\mathcal{K} = \langle w\rangle$ have the required properties.

We were, however, been unable to find even any nilpotent group which has two such subgraphs—nilpotent groups might therefore be the right class of group to look at if one is trying to show that the answer to the above question is negative.

5 Sheehan's fixing subgraphs

The idea of fixing subgraphs was introduced by John Sheehan in [25, 26, 27]. Since then it has turned out that fixing subgraphs are important in many areas of graph theory—an excellent survey of this development is given by [24]. We shall here point briefly to the connection between fixing subgraphs and pseudosimilarity, focusing in particular on a consequence of Theorem 4.

A spanning subgraph U of a graph G is termed a *fixing subgraph* of G if G contains exactly $|\mathrm{Aut}(G)|/|\mathrm{Aut}(G) \cap \mathrm{Aut}(U)|$ subgraphs isomorphic to U (the graph G must contain at least this number). If, in addition, $\mathrm{Aut}(U) \leq \mathrm{Aut}(G)$ then U is called a *strong fixing subgraph* of G. Let $F(G)$ ($F^*(G)$) be the set of fixing (strong fixing) subgraphs of G.

The connection with pseudosimilarity is that if an edge e has a pseudosimilar mate then the spanning subgraph $G-e$ cannot be in $F^*(G)$. As a direct corollary of Theorem 4 Sheehan proves,

Theorem 10 ([24]) *There are infinitely many graphs G such that*

(i) $G - e \notin F^*(G)$ *for all* $e \in E(G)$, *and*

(ii) $|F^*(G)| = 1$.

6 Large sets of mutually pseudosimilar vertices or edges

With the settling of the question of the existence of graphs in which every edge has a pseudosimilar mate, the most interesting and difficult problem in pseudosimilarity would now seem to be the following.

Question 6 *In a graph G of order n, what is the largest possible size k of a set of mutually pseudosimilar vertices? Alternatively, given k, what is the smallest graph which contains k mutually pseudosimilar vertices? What is the answer for the analogous questions on mutually pseudosimilar edges?*

This seems to be a very difficult question. We shall here review some constructions which attempt to pack as many as possible mutually pseudosimilar vertices (or edges) in a graph of order n. It is clear that not all of $V(G)$ can be mutually pseudosimilar, for such a graph G would be regular and an isomorphism from $G-u$ to $G-v$ could therefore be extended to an automorphism of G mapping u into v. With slightly more work one can also show that k must be less than $n-1$. Also, this question has been resolved for trees (in [5] it is shown that $k < 3$ for any tree), for $k = 2$ ([7]; G must have order at least 6) and, it seems, for $k = 3$ (in [13] a graph on 17 vertices with three mutually pseudosimilar vertices is constructed, and this seems to be the smallest possible graph for $k = 3$).

The difficulty of Question 6 and these partial results suggest two questions.

Question 7 *Are there other interesting classes $\mathcal{C}$ of graphs such that, for any graph G in $\mathcal{C}$, the number of mutually pseudosimilar vertices in G must be less than some constant?*

Question 8 *Verify that a graph with $k = 3$ mutually pseusosimilar vertices must have order at least 17. What would be the analogous result for $k = 4$?*

But now we shall be considering sequences of graphs for which k, the number of mutually pseudosimilar vertices, increases without bound.

The simplest way [12] obtain such a sequence is to start with the transitive tournament T_k on k vertices (that is, the tournament with vertex-set $\{1, 2, \ldots, n\}$ in which i dominates j if and only if $i < j$). Clearly the vertices of T_k are all mutually pseudosimilar, but the tournament has to be transformed into an undirected graph while preserving the pseudosimilarity of its vertices. This process is illustrated for T_4 in Figure 2.

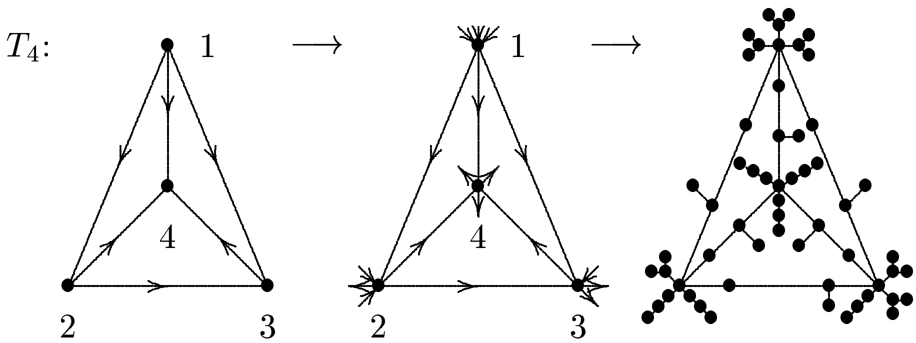

Figure 2: Transforming T_4 into a graph with 4 mutually pseudosimilar vertices

This construction gives a sequence of graphs G_k having k mutually pseudosimilar vertices and order $O(k^2)$.

Another general construction for creating a sequence of graphs with large sets of mutually pseudosimilar vertices runs as follows:

> Let G' be a graph containing r endvertices, all of which are mutually pseudosimilar. Let G be the graph obtained from G' by removing all its endvertices, and let R be the set of neighbours of the endvertices of G'—since no two endvertices are similar, no two can share a common neighbour, therefore $|R| = r$. Let X be the set of all those vertices of G which are in the same orbit as some vertex in R under the action of $\mathrm{Aut}(G)$. We now construct a sequence of graphs G_t, $t = 1, 2, \ldots$, containing r^t mutually pseudosimilar endvertices. Let $G_1 = G'$ and let H_1 be G_1 less one of its endvertices. Having constructed G_t, let H_t be G_t less one of its *pseudosimilar* endvertices. Then, G_{t+1} is obtained by attaching a copy of G_t to each vertex in R and a copy of H_t to each of the other vertices in

$X-R$. (By attaching a copy of G_t (or H_t) to a vertex v of G we mean joining v to every vertex of G_t (or H_t) which is *not* an endvertex.)

Each graph G_t so obtained has r^t mutually pseudosimilar endvertices and $O(|X|^t)$ vertices. Therefore if $k = r^t$ is the number of pseudosimilar endvertices, then the total number of vertices in G_t is $O(k^{\log|X|/\log|R|})$.

(Since the pseudosimilar vertices resulting from this construction are endvertices, that is, vertices of degree 1, the edges incident to these endvertices are also mutually pseudosimilar.)

The crucial step in the above construction is finding the starting graph G', that is, one with endvertices all of which are mutually pseudosimilar. We shall describe different methods which have been employed in order to do this.

Krishnamoorthy and Parthasarathy [16] started with the tournament on three vertices forming a directed cycle. If an endvertex is attached to two vertices of the tournament and the arcs are transformed into edges using "gadgets" as in the proof of Frucht's Theorem, then the two endvertices are pseudosimilar and the resulting graph $G' = G_1$ can be used as the base graph in the above construction. The graph G_2 obtained in this sequence, containing $2^2 = 4$ mutually pseudosimilar vertices, is shown in Figure 3. Starting with this base graph therefore gives a sequence of graphs G_t with $k = 2^t$ mutually pseudosimilar endvertices and order $O(k^{\log 3/\log 2})$.

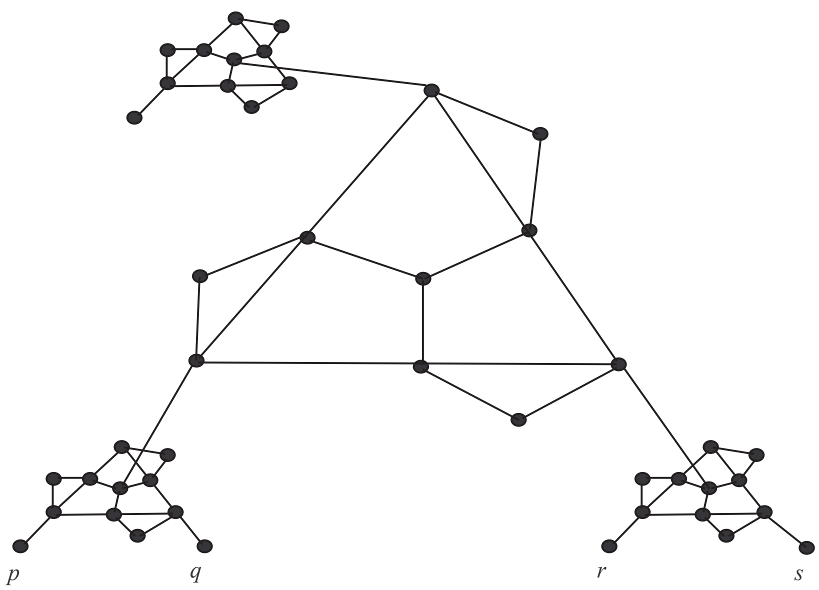

Figure 3: The graph G_2 with 2^2 mutually pseudosimilar vertices p, q, r, s

In [20] a different starting graph was used by exploiting the arc homogeneous property of the quadratic residue tournaments. Thus, consider $QT(7)$, the quadratic residue tournament on seven vertices (that is, the tournament with vertex-set $\{1, 2, \ldots, 7\}$ such that (i, j) is an arc if and only if $j - i$ is a nonzero square modulo 7). The vertices $1, 2, 3$ form a transitive subtournament of $QT(7)$ so that if an endvertex is joined to each of $1, 2, 3$ and the arcs of the tournament are transformed into edges by means of appropriate gadgets, then we get the graph $G' = G_1$ with three endvertices all of which are pseudosimilar. The above construction then yields a sequence of graphs G_t with $k = 3^t$ mutually pseudosimilar endvertices and order $O(k^{\log 7/\log 3})$, which is better than the construction of Kimble, Schwenk and Stockmeyer using transitive tournaments, but not as good as the construction of Krishnamoorthy and Parathasarthy.

The problem of finding a base graph G' as the starting graph of the above construction can be described in terms of permutation groups. Suppose Γ is a group of permutations acting on some set X such that, for some $R \subset X$, the following two conditions hold: (i) the setwise stabiliser $\Gamma_{\{R\}}$ of R is the identity and, (ii) for any two $(|R| - 1)$-subsets A, B of R, there is a permutation α in Γ such that $\alpha(A) = B$. Then, by a result of Bouwer [3], one can construct a graph G with minimum degree at least 2 and $X \subseteq V(G)$ and whose automorphism group is isomorphic to Γ and such that X is invariant under the action of $\mathrm{Aut}(G)$ and also $\mathrm{Aut}(G)$ has the same action as Γ on X. Therefore if we attach one endvertex to each vertex of $R \subset V(G)$ we obtain the starting graph G' all of whose endvertices are mutually pseudosimilar. Hence such starting graphs can be constructed if permutation groups satisfying conditions (i) and (ii) are found.

In [17] such a permutation group with $|X| = 8$ and $|R| = 4$ was constructed. Let Γ be the group of affine transformations on the field $GF(8)$. This group is not 3-transitive but it is 3-homogeneous [22] (that is, any two 3-sets are similar under the action of Γ). Therefore all we need is a 4-set R such that $\Gamma_{\{R\}}$ is trivial. If we represent $GF(8)$ as $\mathbb{Z}_2[x]/p(x)$, where $p(x)$ is the primitive, irreducible (over $\mathbb{Z}_2$) polynomial $x^3 + x + 1$, and if we let $R = \{0, 1, x, x^2\}$, then one can easily check that the only permutation in Γ which fixes R setwise is the identity.

This then gives a starting graph G' with 4 endvertices all mutually pseudosimilar, and therefore a sequence of graphs G_t with $k = 4^t$ mutually pseudosimilar endvertices and order $O(k^{3/2})$. Till now, this sequence seems to be the one which gives the best "packing" of mutually pseudosimilar vertices.

In [17] there is also described a construction which produces, for all r, a graph containing r endvertices *all* of which are mutually pseudosimilar. However, this construction requires that $|X| = O(|R|^{2|R|})$ and it therefore does not solve the problem of obtaining as dense a packing of mutually pseudosimilar vertices as possible.

In [17] it is also shown that a permutation group satisfying Conditions (i) and (ii) above must have $|X| \geq 2|R| - 1$. Therefore the above construction can, at best, produce a sequence of graphs G_t with $k = r^t$ mutually pseudosimilar endvertices and order $O(k^{\log(2r-1)/\log r})$.

The above constructions suggest the following questions, the first two of which are restricted versions of Question 6. In view of the preceding comments, a positive answer to Question 9 would require a totally different construction from the one we have been discussing. The constructions used in [8, 13] employ Cayley graphs and exploit the equivalence of the action of a permutation group Γ on a set X with its action on the set of cosets of a stabiliser. In [15], Kocay, Niesink and Zarnke systematically search for groups Γ with a subgroup $\mathcal{K}$ such that the action of Γ on the cosets of $\mathcal{K}$ can be used to construct graphs with $4 \geq k \geq 2$ pseudosimilar vertices. Perhaps these methods need to be investigated and extended further in order to tackle this problem.

Question 9 *Is it possible to construct a sequence of graphs $\langle G_k \rangle$ such that G_k has k mutually pseudosimilar vertices and order $O(k)$?*

Question 10 *Given k, what is the smallest graph which contains k endvertices* **all** *of which are mutually pseudosimilar?*

The next question asks whether there are tournaments which extend the arc homogeneous property of the quadratic residue tournaments to a type of local homogeneity with respect to one of its subtournaments. Such tournaments could be used (as the tournament $QT(7)$ was used above) in order to obtain the base graph G' for the above construction.

Question 11 *Can one construct, for any $k \geq 4$, a tournament A_k with the following property: A_k contains, as a subtournament, a transitive tournament T_k on k vertices such that, for any two subtournaments T_{k-1} and T'_{k-1} of T_k on $k-1$ vertices, there is an automorphism α of A_k such that $\alpha(T_{k-1}) = T'_{k-1}$.*

Finally, one can ask questions analogous to Question 1, namely whether there is a characterisation similar to Theorem 1 of graphs with $k > 2$ mutually pseudosimilar vertices. In [14] a theorem analogous to Theorem 1 was proved, but there the graph H could be infinite. In [8] this problem was partially solved for $k = 3$ with the extra assumption that there are no edges between a certain set of vertices containing the pseudosimilar ones. One can therefore ask the following.

Question 12 *Suppose a graph G has $k > 2$ mutually pseudosimilar vertices $u_1, u_2, \ldots, u_k$. Is G the induced subgraph of a finite graph H in which $u_1, \ldots, u_k$ are similar and which has $k-1$ automorphisms $\alpha_1, \ldots, \alpha_{k-1}$ such that $\alpha_i(G - u_1) = G - u_{i+1}$ and such that the vertices in $V(H) - V(G)$ are in the same orbit as $u_1, \ldots, u_k$ under the action of $Aut(H)$?*

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