Cayley graphs, pseudosimilar edges, and line-graphs

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Abstract

Two vertices in a graph H are said to be pseudosimilar if $H-u$ and $H-v$ are isomorphic but no automorphism of H maps u into v. Pseudosimilar edges are analogously defined. Graphs in which every vertex is pseudosimilar to some other vertex have been known to exist since 1981. Producing graphs in which every edge is pseudosimilar to some other edge proved to be more difficult. We here look at two constructions of such graphs, one from $\frac{1}{2}$-transitive graphs and another from edge-transitive but not vertex-transitive graphs. Some related questions on Cayley line-graphs are also discussed.

1 Introduction

Two vertices u and v in a graph H are said to be *similar* if H has an automorphism that maps u into v; they are called *removal-similar* if $H-u$ and $H-v$ are isomorphic, and *pseudosimilar* it they are removal-similar but not similar. Analogous definitions hold for similar, removal-similar and pseudosimilar edges. For a survey on pseudosimilarity the reader is referred to [7]

A permutation group is said to act *regularly* on a finite set if its action on the set is transitive and fixed-point free. If the automorphism group $\mathrm{Aut}(G)$ of a graph G acts regularly on $V(G)$ then we say that G is a *graphical regular representation* (GRR) of $\mathrm{Aut}(G)$.

Let Γ be a group and let $S \subseteq \Gamma$ be such that $\Gamma = \langle S \rangle$, $1 \notin S$ and $S^{-1} = S$, where $S^{-1} = \{s^{-1} : s \in S\}$. A graph G is a *Cayley graph* of G with respect to S if $V(G) = \Gamma$ and u, v are adjacent in G if and only if $v = us$ for some $s \in S$. We denote this situation by writing $\mathrm{Cay}(\Gamma, S)$ for G.

It is known that $G = \mathrm{Cay}(\Gamma, S)$ for some Γ, S if and only if Γ is isomorphic to a subgroup of $\mathrm{Aut}(G)$ acting regularly on the vertex-set of G. In particular, a graph G is a GRR of $\mathrm{Aut}(G)$ if and only if G is a Cayley graph of its automorphism group $\mathrm{Aut}(\Gamma)$.

In [6], graphs H are constructed in which every vertex has a pseudosimilar mate—that is, for every vertex $v \in V(H)$ there exists a $v' \in V(H)$ such that v and v' are pseudosimilar. The construction briefly runs as follows. Let G be a GRR of its automorphism group $\mathrm{Aut}(G)$ and let the order of $\mathrm{Aut}(G)$ be odd. (Such a group must be nonabelian. Moreover, such graphs and groups do exist [4].) Let r be a vertex of G, and let $H = G - r$. Since G is a GRR, H has the trivial automorphism group, therefore no two vertices of H are similar. Also, for every vertex v of H there is an $\alpha \in \mathrm{Aut}(G)$ and a vertex v' of H such that $\alpha(v) = r$ and $\alpha(r) = v'$, Therefore $H - v \simeq H - v'$. Moreover, $v \neq v'$ since otherwise $\alpha^2(r) = r$ and this is impossible because $|\mathrm{Aut}(G)|$ is odd. Hence the vertices of H are paired by pseudosimilarity, as required. We shall call this the Kimble-Schwenk-Stockmeyer (KSS) construction.

Kimble, Schwenk and Stockmeyer also asked if graphs can be found in which all edges are paired by pseudosimilarity. This question was answered in the affirmative in [8] where the following result was proved.

Theorem 1 *There are infinitely many graphs H such that, for every edge e of H, there is an edge e' which is pseudosimilar to e.*

We shall here look more closely at the smallest graph given by the above theorem, and we shall also construct another example of a graph all of whose edges are pseudosimilar which is not amongst those given by this theorem. We shall also discuss relationships with Cayley line-graphs.

2 Construction from $\frac{1}{2}$-transitive graphs

In [8] the KSS construction was adapted as follows in order to obtain pseudosimilar edges. Let G be a graph with an odd number of edges and whose automorphism group acts regularly on its edges—then the graph $H = G - e$, for any edge e, would have all of its edges paired by pseudosimilarity.

The graphs G with these properties which were used in [8] had been constructed in [1]. A special case of this family of graphs can be described as Cayley graphs in the following way.

Let p be a prime and $p \equiv 1 \pmod 3$ and $p \equiv 1 \pmod 5$. Let Γ be the group defined by

$$\Gamma = \langle b, c | b^3 = c^p = 1, c^b = b^{-1}cb = c^r \rangle$$

where r is such that $r^3 = 1 \pmod 3$. Let t be such that $t^5 = 1 (\bmod\ p)$, and let a be the automorphism of Γ defined by $b^a = b$ and $c^a = c^t$. Let

$$T = \{c^a, c^{a^2}, c^{a^3}, c^{a^4}, c^{a^5} = c\}$$

and

$$S = bT \cup T^{-1}b^{-1} = bT \cup T^{-1}b^2 = S_1 \cup S_2.$$

If G is the Cayley graph $\mathrm{Cay}(\Gamma, S)$, then it is shown in [1] that the automorphism group of G is acts regularly on its edges (in fact, the automorphism group is equal to the relative holomorph $\mathrm{Hol}(\Gamma, a)$). Since G has order $3p$ and degree 10, it has an odd number, $15p$, of edges, and so we obtain a graph with our required properties. All these graphs are $\frac{1}{2}$-transitive, that is, the automorphism group is transitive on the vertices and the edges, but not on the directed edges.

The smallest value of p for which the above construction works is $p = 31$, giving a graph H with 464 edges, all of them paired by pseudosimilarity. We shall here give a direct proof that the corresponding graph G with 465 edges has the property that its automorphism group acts regularly on its edges—that is, the group acts transitively on the edges and the stabiliser of any edge is trivial.

Edge transitivity is easy: The left regular translation by $(bc^a)^{-1}$ maps the edge $\{1, bc^a\}$ into the edge $\{1, (bc^a)^{-1}\}$ while the automorphism a^i (which fixes the set S and is therefore also a graph automorphism) maps the edge $\{1, bc\}$ into the edge $\{1, bc^{a^i}\}$. We now need to show that only the trivial automorphism of G fixes an edge.

We note that any edge $\{x, xs\}$, $s \in S$, is on one and only one triangle: $\{x\ (= xs^3), xs, xs^2\}$. This is because each element of S has order 3 and because no three elements of S are such that $s_1s_2s_3 = 1$ unless $s_1 = s_2 = s_3$. This means that any automorphism which fixes the edge $\{x, xs\}$ (that is, either fixes or transposes the vertices x and xs) must also fix the vertex xs^2.

From now on let $p = 31$. We shall take $r = 25$ and $t = 2$. Then, modulo 31, $r^2 = 5$, $r^3 = 1$, and $\{t, t^2, \ldots, t^5\} = \{2, 4, 8, 16, 1\}$. Also, S_1 is the set of all elements bc^{t^i} and S_2 is the set of all elements $b^2c^{-r^2t^i}$.

Consider, without loss of generality, the edge $\{1, bc\}$. On how many 4-cycles does it lie? This edge lies on a 4-cycle if there are elements $s_1, s_2, s_3 \in S$ such that $1 = bcs_1s_2s_3$, and such that no two (cyclically) consecutive elements in the product are inverses. It can be checked that this can happen only if $s_1, s_2 \in S_2$ and $s_3 \in S_1$. Therefore we have

$$1 = bc \cdot b^2c^{-r^2t_1} \cdot b^2c^{-r^2t_2} \cdot bc^{t_3}$$

and, comparing powers of c, this gives that

$$r^2(t_1 - 1) = t_3 - t_2 \pmod{31}.$$

Checking all possibilities shows that there are only two solutions,

$$t_1 = 8,\ t_2 = 4,\ t_3 = 8$$

and

$$t_1 = 4,\ t_2 = 1,\ t_3 = 16.$$

These considerations enable us to draw the subgraph of G induced by the vertex 1, its neighbours, and all the 4-cycles containing edges incident to 1. Let this subgraph be denoted by G_1. In other words, G_1 is made up of all 3-cycles and 4-cycles containing the vertex 1. This graph is shown in Figure 1, with the names of some of the vertices showing.

Note that, for any vertex v, the corresponding subgraph G_v is obtained from Figure 1 by pre-multiplying every label by v.

Now suppose that the edge $\{bc, (bc)^{-1}\}$ is fixed by some automorphism α of G. Therefore the vertex 1 is fixed, and so is the subgraph G_1. The automorphism α therefore either fixes all the vertices of G_1 or else it induces an involution on $V(G_1)$ which transposes bc and $(bc)^{-1}$, bc^{t^4} and $(bc^t)^{-1}$, etc, and it fixes c^{23} and c^4 (cf. Figure 1). Suppose, for contradiction, that the latter holds, that is, α does not fix all vertices of G_1.

Now, consider Figure 2, which depicts G_{c^4} with the names of a few of the vertices showing. Since c^4 and 1 are fixed by α, then either c^{27} is fixed or it is interchanged with c^2. But, from Figure 1, c^{27} is interchanged with c^{15}, giving us the required contradiction. Therefore all the vertices of G_1 are fixed by α. Repeating this argument starting from G_x for $x \in G_1$ finally gives, since G is connected, that α fixes all vertices of G.

3 Construction from non-vertex-transitive graphs

One question which arises following the previous construction is whether or not it is possible to have a graph G whose automorphism group acts regularly on its edges but not transitively on the vertices. Such graphs would, of course, have to be bipartite. We shall construct an example using the same group of order 3.5.31 as in the previous section. But first we give some general results. The motivating idea behind these is the well-known characterisation, due to Sabidussi [9], of vertex transitive graphs in terms of coset graphs.

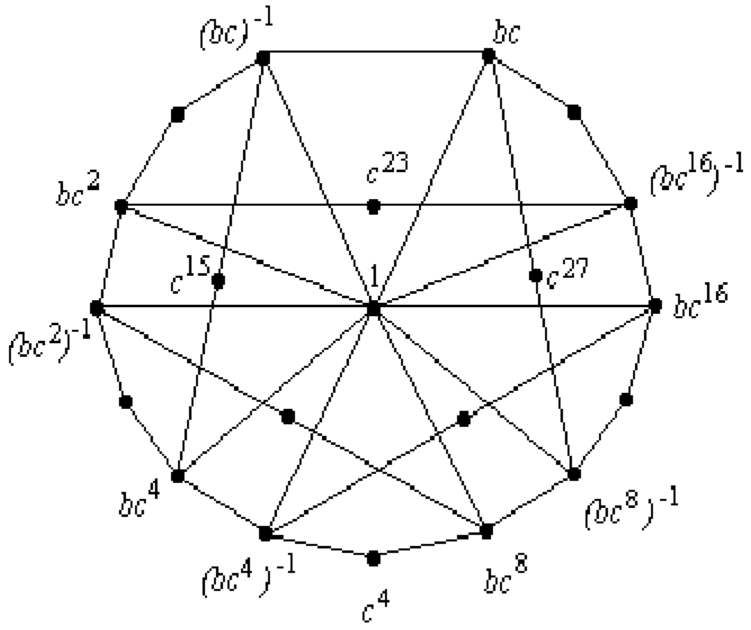

Figure 1: The subgraph G_1

Let Γ be a group and $\mathcal{H}, \mathcal{K}$ two subgroups of Γ. Let S be a subset of Γ. Define the graph $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K}, S)$ as follows: Its vertices are the left cosets of $\mathcal{H}$ and of $\mathcal{K}$; two cosets $x\mathcal{H}$ and $y\mathcal{H}$ are adjacent if and only if $y^{-1}x \in \mathcal{K}S\mathcal{H}$. If, moreover, $S \subseteq \mathcal{K}\mathcal{H}$, that is, $\mathcal{K}S\mathcal{H} = \mathcal{K}\mathcal{H}$, then we denote $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K}, S)$ simply by $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$.

If $\mathcal{H} \cap \mathcal{K} = \{1\}$, then any two cosets $x\mathcal{H}$, $y\mathcal{K}$ are either disjoint or have exactly one element in common. In this case, $x\mathcal{H}$ and $y\mathcal{K}$ are adjacent in $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ if and only if they are not disjoint, that is, all edges of $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ are of the form $\{t\mathcal{H}, t\mathcal{K}\}$, where t is the element common to both cosets. Another useful way to look at adjacencies in $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ when $\mathcal{H} \cap \mathcal{K}$ is trivial is as follows: The coset $x\mathcal{H}$ is adjacent to all the

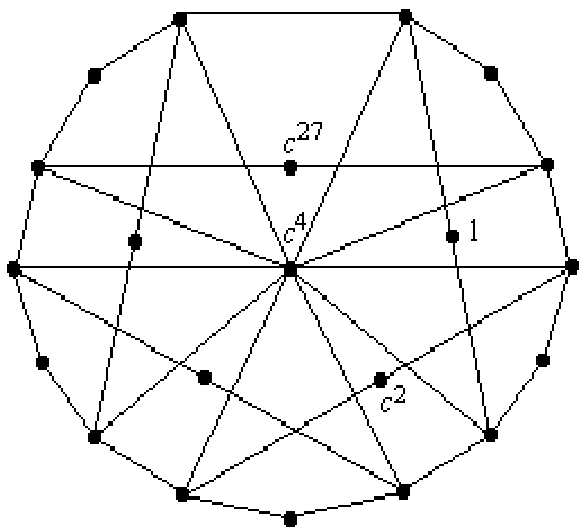

Figure 2: The subgraph G_{c^4}

cosets $xh\mathcal{K}$, for all $h \in \mathcal{H}$ (all these cosets are distinct); similarly, the coset $y\mathcal{K}$ is adjacent to all the cosets $yk\mathcal{K}$ for all $k \in \mathcal{K}$. Clearly, the degrees of the cosets $x\mathcal{H}$ and $y\mathcal{K}$ as vertices in $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ are $|\mathcal{H}|$ and $|\mathcal{K}|$, respectively.

The following two results are not difficult to prove.

Theorem 2 *Let G be a graph whose vertex-set is partitioned into two orbits V_1, V_2 under the action of the automorphism group Γ. Let $\mathcal{H}$ be the stabiliser of the vertex $u \in V_1$ and $\mathcal{K}$ the stabiliser of the vertex $v \in V_2$. Let S be the set of all those permutations $\alpha \in \Gamma$ such that $\alpha(u)$ is adjacent to v. Then G is isomorphic to $Cos(\Gamma, \mathcal{H}, \mathcal{K}, S)$. Moreover, if G is edge-transitive then $S \subseteq \mathcal{K}\mathcal{H}$, that is, G is isomorphic to $Cos(\Gamma, \mathcal{H}, \mathcal{K})$.*

Theorem 3 *Let $G = Cos(\Gamma, \mathcal{H}, \mathcal{K})$. For $t \in \Gamma$, let λ_t denote the action of left translation by t on the left cosets of $\mathcal{H}$ and $\mathcal{K}$. Then λ_t is an automorphism of G; this action is transitive on the edges of G. Suppose ϕ is an automorphism of Γ which fixes setwise both $\mathcal{H}$ and $\mathcal{K}$. Let $\hat{\phi}$ denote the induced action on the cosets of $\mathcal{H}$ and $\mathcal{K}$. Then $\hat{\phi}$ is an automorphism of the graph G.*

From these two theorems it is clear that to obtain a graph whose automorphism group acts regularly on the edges but non-transitively on the vertices we need to find a coset graph $\mathrm{Cos}(\Gamma, \mathcal{H}, \mathcal{K})$ such that no automorphism of the group fixes $\mathcal{H}$ and $\mathcal{K}$. Of course we also require the graph to be connected, therefore $\mathcal{H} \cup \mathcal{K}$ must generate all of Γ.

We shall see that the group of order $3 \cdot 5 \cdot 31$ which we encountered in the previous section will also work for us here. This group Ξ will be the relative holomorph with the automorphism a of the group Γ of order $3 \cdot 31$ of Section 3. That is

$$\Xi = \langle a, b, c | a^5 = b^3 = c^{31} = 1, ba = ab, ca = ac^2, cb = bc^{25} \rangle.$$

In the following it will sometimes be convenient to let $w = bc$ and to present Ξ as

$$\Xi = \langle a, w, c | a^5 = w^3 = c^{31} = 1, wa = awc, ca = ac^2, cw = wc^{25} \rangle.$$

Now let $\mathcal{H}$ be the cyclic subgroup generated by a and let $\mathcal{K}$ be the cyclic subgroup generated by w. Let $G = \mathrm{Cos}(\Xi, \mathcal{H}, \mathcal{K})$. This graph is edge-transitive but not vertex-transitive since the cosets of $\mathcal{H}$ have degree 5 whereas the cosets of $\mathcal{K}$ have degree 3. Moreover, it is not difficult to check that no nontrivial automorphism of the group Ξ fixes $\mathcal{H} \cup \mathcal{K}$, therefore there is reason to hope that, in fact, the full automorphism of G is Ξ, that is, the automorphism group of G acts regularly on the edges. This we now proceed to prove.

Our first step is to determine the girth of G. Let e be the edge $\{\mathcal{H}, \mathcal{K}\}$. The vertices of any cycle of length $2l$ passing through the edge e form a sequence

$$\mathcal{H},\ \mathcal{K},\ w^{i_1}\mathcal{H},\ w^{i_1}a^{j_1}\mathcal{K},\ \ldots\ ,\ w^{i_1}a^{j_1}\ldots w^{i_{l-1}}a^{j_{l-1}}w^{i_l}\mathcal{H} = \mathcal{H}$$

where the powers of w and a are not equal to zero modulo 3 and modulo 5, respectively. It is not difficult to check by hand that, for $l \leq 3$, there is no solution of the equation

$$w^{i_1}a^{j_1}\ldots w^{i_{l-1}}a^{j_{l-1}}w^{i_l}\mathcal{H} = \mathcal{H}$$

for nonzero powers of the w and a. For $l = 4$ this equation becomes

$$w^{i_1}a^{j_1}w^{i_2}a^{j_2}w^{i_3}a^{j_3}w^{i_4}\mathcal{H} = \mathcal{H}. \tag{1}$$

Since $(bc)^2 = b^2c^{26}$,

$$w^{i_1}a^{j_1}w^{i_2}a^{j_2}w^{i_3}a^{j_3}w^{i_4} = a^{j_1+j_2+j_3}b^{i_1+i_2+i_3+i_4}c^k$$

where the power k of c is equal to

$$s_{i_1}2^{j_1+j_2+j_3}25^{i_2+i_3+i_4} + s_{i_2}2^{j_2+j_3}25^{i_3+i_4} + s_{i_3}2^{j_3}25^{i_4} + s_{i_4}$$

where s_i is defined by $s_1 = 1$ and $s_2 = 26$.

Therefore Equation 1 implies that

$$i_1 + i_2 + i_3 + i_4 = 0 \pmod 3 \tag{2}$$

and

$$s_{i_1}2^{j_1+j_2+j_3}25^{i_2+i_3+i_4} + s_{i_2}2^{j_2+j_3}25^{i_3+i_4} + s_{i_3}2^{j_3}25^{i_4} + s_{i_4} \\ = 0 \pmod{31}. \tag{3}$$

A computer search revealed that the only solutions of Equations 2 and 3 for $(i_1, j_1, i_2, j_2, i_3, j_3, i_4)$ are

$$(1,3,2,2,2,1,1),\ (1,3,1,2,2,1,2),\ (1,4,1,3,2,2,2),\ (1,2,2,1,2,4,1)$$

$$(2,1,2,4,1,3,1),\ (2,1,1,4,1,3,2),\ (2,4,1,3,1,2,2),\ (2,2,2,1,1,4,1).$$

This means that the girth of G is 8 (which is the largest possible given its order and the degrees of its vertices) and that through the edge $\{\mathcal{H}, \mathcal{K}\}$ (and hence through any edge) there are exactly eight cycles of length 8. A consideration of the above eight solutions, and the fact that G is edge-transitive, leads to Figure 3, which shows all the 8-cycles passing through any of the three edges incident to $\mathcal{K}$. In this figure, the names of some of the vertices is given, and these indicate some of the solutions given above. We note, in particular, the vertices $a\mathcal{K}$ and $a^3\mathcal{K}$. The vertex $a\mathcal{K}$ is equal to both $wa^3w^2a^2w^2a\mathcal{K}$ and $wa^3wa^2w^2a\mathcal{K}$ since $wa^3w^2a^2w^2a = aw^2$ and $wa^3wa^2w^2a = aw$ (cf. the first two solutions given above). The vertex $a^3\mathcal{K}$ is similarly worked out since $w^2aw^2a^4wa^3 = a^3w^2$ and $w^2awa^4wa^3 = a^3w$ (cf. the first two solutions in the second row above). (The *Mathematica* package was extensively used to carry out all the above calculations.)

Figure 3 therefore gives all the 8-cycles passing through the three edges $\{\mathcal{K}, \mathcal{H}\}, \{\mathcal{K}, w\mathcal{H}\}, \{\mathcal{K}, w^2\mathcal{H}\}$. We note that the corresponding figure for the three edges $\{t\mathcal{K}, t\mathcal{H}\}, \{t\mathcal{K}, tw\mathcal{H}\}, \{t\mathcal{K}, tw^2\mathcal{H}\}$ incident to the vertex $t\mathcal{K}$ can be obtained from Figure 3 by pre-multiplying every label by t. Therefore, from Figure 3, if an automorphism of G maps the edge $\{t\mathcal{H}, t\mathcal{K}\}$ into the edge $\{r\mathcal{H}, r\mathcal{K}\}$, then this automorphism must map the pair of edges $\{t\mathcal{H}, ta\mathcal{K}\}, \{t\mathcal{H}, ta^3\mathcal{K}\}$ into the pair $\{r\mathcal{H}, ra\mathcal{K}\}, \{r\mathcal{H}, ra^3\mathcal{K}\}$.

Now suppose that α is an automorphism of G which fixes the edge $\{\mathcal{H},\mathcal{K}\}$. Suppose also, for contradiction, that α transposes the vertices $w\mathcal{H}$ and $w^2\mathcal{H}$. Therefore it must transpose the vertices $a\mathcal{K}$ and $a^3\mathcal{K}$, that is, it transposes the edges $\{\mathcal{H},a\mathcal{K}\}$ and $\{\mathcal{H},a^3\mathcal{K}\}$. Therefore, by the previous observation, α must transposes the two sets of edges $\{\{\mathcal{H},a^2\mathcal{K}\},\{\mathcal{H},a^4\mathcal{K}\}\}$ and $\{\{\mathcal{H},a^4\mathcal{K}\},\{\mathcal{H},a\mathcal{K}\}\}$. But this is impossible since the edge $\{\mathcal{H},a\mathcal{K}\}$ is transposed with the edge $\{\mathcal{H},a^3\mathcal{K}\}$.

Therefore the vertices $w\mathcal{H}$ and $w^2\mathcal{H}$ must be fixed by α. Hence we have that, if α fixes the edge $\{\mathcal{H},\mathcal{K}\}$ then it must also fix the edges $\{\mathcal{K},w\mathcal{H}\}$, $\{\mathcal{K},w^2\mathcal{H}\}$, $\{w\mathcal{H},wa\mathcal{K}\}$ and $\{w^2\mathcal{H},w^2a^3\mathcal{K}\}$ (cf. Figure 3).

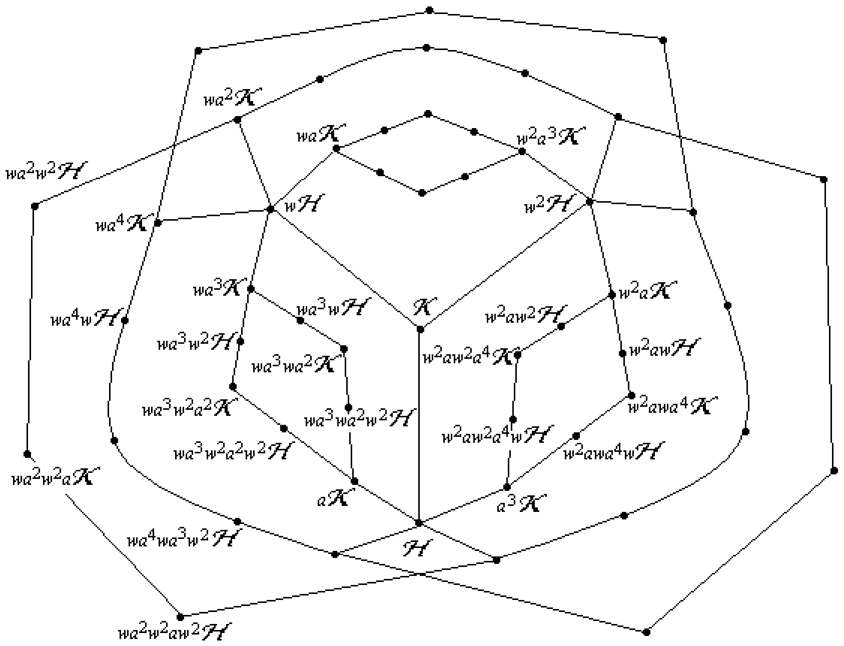

Figure 3: All the 8-cycles passing through the edges incident to $\mathcal{K}$

Repeating this argument starting from the edge $\{w\mathcal{H},wa\mathcal{K}\}$ leads to the conclusion that α fixes all the edges of G, that is, it is the trivial automorphism.

4 Cayley line-graphs

The problem of finding a graph whose automorphism group acts regularly on its edges can be seen as the problem of finding a GRR which is also a line-graph. We are therefore looking for a line-graph G with an odd number of vertices and which is a GRR of its automorphism group.

Such a graph G must be a Cayley graph $\mathrm{Cay}(\Gamma, S)$. Also, G is a line-graph if and only if its edge-set can be partitioned into complete subgraphs such that no vertex lies in more than two of the subgraphs [3]. We therefore require, apart from the condition $S = S^{-1}$, that S can be partitioned as $S_1 \cup S_2$ such that $s^{-1}t \in S$ if and only if both s and t are in the same S_i. One way, therefore, to ensure that the Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is a line graph would be to let S be $\mathcal{H} \cup \mathcal{K} - \{1\}$ such that $\mathcal{H}, \mathcal{K}$ are subgroups of Γ and $\mathcal{H} \cap \mathcal{K} = \{1\}$.

Now, for the Cayley graph G to be a GRR it is necessary that there be no nontrivial automorphism ϕ of Γ such that $\phi(S) = S$. This necessary condition is not, in general, sufficient. The following result of Godsil [2], however, affirms that for a special class of p-groups this simple condition is, in fact, sufficient to guarantee that the Cayley graph is a GRR.

Theorem 4 *[2] Let G be a finite p-group which admits no homomorphism onto the wreath product of Z_p by Z_p. Let $S \subset G, S = S^{-1}, G = \langle S \rangle$ such that no nontrivial automorphism of G fixes S setwise. Then the Cayley graph of G with respect to S is a GRR of G.*

Godsil's theorem and the above discussion led Lauri and Scapellato [8] to pose the following question:

Problem *Does there exist a p-group Γ (p an odd prime) having two subgroups $\mathcal{H}, \mathcal{K}$ with the following properties: (i) $\mathcal{H} \cap \mathcal{K} = \{1\}$, (ii) $\Gamma = \langle \mathcal{H} \cup \mathcal{K} \rangle$, and (iii) no nontrivial automorphism of Γ fixes $\mathcal{H} \cup \mathcal{K}$ setwise?*

(We have, in fact, been unable to find any nilpotent group which has two subgraphs $\mathcal{H}$, $\mathcal{K}$ with the above three properties. Nilpotent groups might be the right class of group to look at if one is trying to show that the answer to the above question is negative.)

We finally note that the above is not the only way for a Cayley graph $\mathrm{Cay}(\Gamma, S)$ to be a line-graph. For, if $\mathcal{H} < \Gamma$, $g \in \Gamma$ such that $g^{-1}\mathcal{H}g \cap \mathcal{H} = \{1\}$, and if $S_1 = \mathcal{H}^* \cup \mathcal{H}g$ and $S_2 = g^{-1}\mathcal{H}^*g \cup g^{-1}\mathcal{H}$, then $\mathrm{Cay}(\Gamma, S_1 \cup S_2)$ is a line graph (here, $\mathcal{H}^*$ denotes $\mathcal{H} - \{1\}$).

In fact, the line-graph of the Cayley graph in Section 2 is the Cayley graph $\mathrm{Cay}(\Xi, S)$ (where Ξ is the group of Section 3) with $S = \mathcal{H}^* \cup \mathcal{H}g \cup g^{-1}\mathcal{H}^*g \cup g^{-1}\mathcal{H}$ where $\mathcal{H} = \langle a \rangle$ and $g = w = bc$.

5 Concluding remarks

Apart from the above problem on p-groups, the constructions discussed here also lead to the following question. All known graphs having all vertices (edges) paired by pseudosimilarity have been obtained by means of the KSS construction, that is, by deleting a vertex (edge) from a graph whose automorphism group acts regularly on its vertices (edges). In other words, a graph obtained by these methods all of whose vertices (edges) are paired by pseudosimilarity can be changed into a vertex-transitive (edge-transitive) graph by the addition of a single vertex (edge).

Is this always the case, or do there exist graphs all of whose vertices (edges) are pseudosimilar but which cannot be obtained by means of the KSS construction?

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