On a determinant formula for enumerating Euler trails in a class of digraphs *

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Abstract

In 1996 Macris and Pulé [6] obtained a new determinant formula for the number of Euler trails in a special class of digraphs. An elementary combinatorial proof of this result was given in [5]. In this note we shall discuss possible extensions and generalisations of this result. Many of these observations are, as yet, in a very inconclusive state and they are being put forward here in the spirit of a workshop session in the hope that some participants might find ways of bringing some of these ideas to fruition.

1 Introduction

A determinant formula for the number of Euler trails in a digraph has been known for several years. This is the well-known BEST formula, named after de-Bruijn and van Aardenne-Ehrenfest [3] and Smith and Tutte [8]. The BEST formula says that if D is a digraph with vertices $\{1, 2, \ldots, n\}$ such that the vertex i has outdegree equal to indegree equal to d_i, ad if K is the principal submatrix obtained from the Laplacian of D obtained by deleting any row and the corresponding column, then the number of Euler trails of D, $Eu(D)$, is given by

$$(d_1 - 1)!(d_2 - 1)! \ldots (d_n - 1)! \det(K)$$

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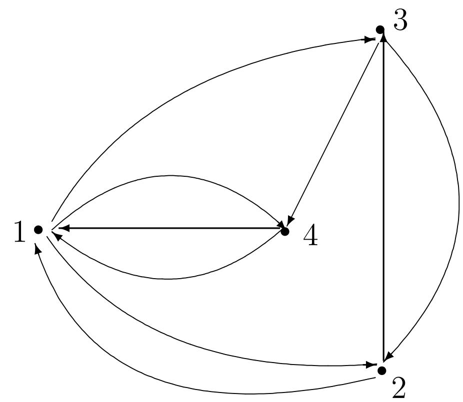


Figure 1: An example of an Eulerian digraph

As an example of the use of this formula consider the digraph shown in Figure 1.

The Laplacian of this digraph is:

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -2 & 0 & 0 & 2 \end{pmatrix}$$

Therefore, using the BEST formula, the number of Euler trails is $2!.1!.1!.1!.6 = 12$.

2 An alternative formula for a special type of digraph

In 1996 Macris and Pulé came up with an alternative determinant formula for digraphs in which each indegree and outdegree equals 2 (which we call *2-in-2-out* digraphs); although they derived their formula from the BEST formula, it seems to have very interesting combinatorial implications of its own.

Consider, a 2-in-2-out digraph D and let γ be any Euler trail in D. Draw on a circle the vertices in the order as they appear traversing the trail γ and number them consecutively from 1 to n (each label will appear twice on the circle). Join by chords common labels and construct the *intersection matrix*

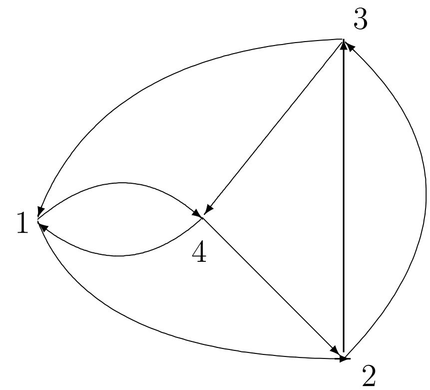


Figure 2: An example of a 2-in-2 out digraph

I_γ as follows: for $i > j$ the ij term of I_γ is 1 if the chords i and j intersect, 0 otherwise. The diagonal terms are 0 and, for $j > i$ the ji term is -1 times the ij term. The matrix I_γ is therefore skew-symmetric. The formula of Macris and Pulé is given by the following theorem.

Theorem 1.

$$Eu(D) = \det(I + I_\gamma)$$

where I is the $n \times n$ identity matrix.

The use of this formula is illustrated by the digraph shown in Figure 2 and one of its Euler trails represented on a circle as shown in Figure 3.

The intersection matrix I_γ corresponding to this Euler trail γ is given by

$$I_\gamma = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Therefore the number of Euler trails in D is given by $\det(I + I_\gamma) = 6$.

In [5] an elementary combinatorial proof of Theorem 1 was given. The main combinatorial idea of this proof was a connection between Euler trails in 2-in-2-out digraphs and determining when the product of tanspositions with a cyclic permutation gives again a cyclic permutation. At each vertex, the Euler trail γ takes one of two possible routes. That is, if vertex i is incident from arcs a, b and incident to arcs c, d, and the trail γ traverses these arcs in the order $\ldots ac \ldots bd \ldots$, then the route which follows these arcs in the order

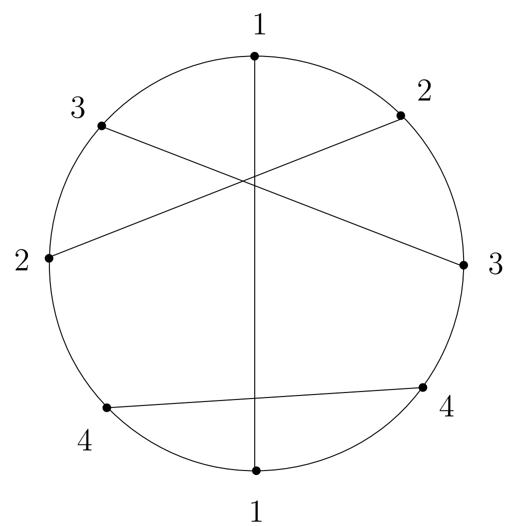


Figure 3: The chords corresponding to an Euler trail γ in the digraph of Figure 2

$\ldots ad \ldots bc \ldots$ will be called the *alternative route* at i. The question is, do we still get an Euler trail if we take the alternative route at some set of vertices? This question is intimately connected with the following question: If $\sigma_1 \ldots \sigma_t$ are disjoint transpostions and τ is the cyclic permutation $(12 \ldots 2n)$, is the product $\sigma_1 \ldots \sigma_t \tau$ still a cyclic permutation? If τ is represented by a circle with the points $1, 2, \ldots, 2n$ placed consecutively round the circle and each transposition is represented by a chord joining the two points it transposes, and if we interpret this diagram as one coming from a 2-in-2-out digraph as described above, then this question is equivalent to the question of whether or not taking the alternative route at each of the vertices corresponding to the chords still yields an Euler trail. The most basic situation arises when $t = 2$, equivalently when the alternative route is taken at exactly two vertices. Only when the chords intersect does a new Euler trail arise (equivalently, $\sigma_1 \sigma_2 \tau$ is a cycle). The new Euler trail (new cycle) which arises is represented in Figure 4.

Now this problem of multiplying a cycle by transpositions had been studied by Cohn and Lempel [2]. They defined a matrix M similar to our matrix I_γ with the difference that all entries were $+1$ since they were working modulo 2. Their main result which interests us here is the following.

Theorem 2. *The product $\sigma_1 \ldots \sigma_t \tau$ is a cycle if and only if* $\det(M) =$

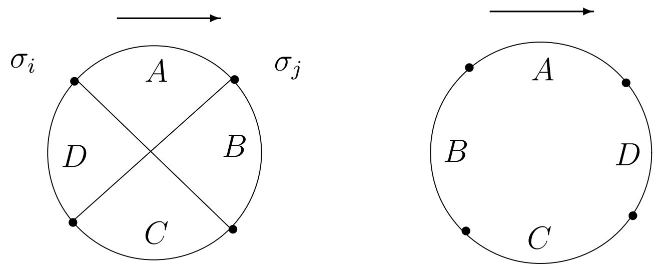


Figure 4: Representations of τ and $\sigma_i\sigma_j\tau$

1 mod 2.

The main theorem in [5] is the following. (Here, if A is an $n \times n$ matrix and $S \subseteq \{1, 2, \ldots, n\}$, then $A[S]$ denotes the principal submatrix of A obtained by taking the rows and columns indexed by S.)

Theorem 3. *The determinant of the intersection matrix I_γ is equal to either 0 or 1.*

Theorem 3 and the preceding comments clinches the result of Macris and Pulé, because let $C(S)$ be the set of Euler trails obtained from γ by taking the alternative route at the vertices in S and only those. Then clearly $|C(S)|$ equals 0 or 1. And $|C(S)| = 1$ occurs if and only if $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_s}\tau$ is a cycle. From the Theorem 3 it follows that $\det(I_\gamma[S])$ equals $\det(M[S])$ mod 2, therefore $|C(S)| = 1$ if and only if $\det(I_\gamma[S]) = 1$. That is, $\det(I_\gamma[S]) = |C(S)|$.

We now use the fact that, if I is the $n \times n$ identity matrix, and X is an $n \times n$ matrix then

$$\det(I + X) = \sum_{\emptyset \subseteq S \subseteq N} \det(X[S]) \tag{1}$$

So now, the number of Euler trails in D is given by

$$\begin{aligned} &= \left| \bigcup_{\emptyset \subseteq S \subseteq N} C(S) \right| \\ &= \bigcup_{\emptyset \subseteq S \subseteq N} |C(S)| \\ &= \sum_{\emptyset \subseteq S \subseteq N} \det(I_\gamma[S]) \\ &= \det(I + I_\gamma) \end{aligned}$$

by Eq. (1).

It is well to point out here that for the proof of Theorem 3 a slightly more general definition of the intersection matrix I_γ is required. The proof of this theorem involves induction which is based on removing a pair of intersecting chords as shown in Figure 4 (see [5] for details). When the cycle is transformed as shown in this figure, it can very well happen that the chords are no longer numbered consecutively. To take care of this situation the matrix I_γ has to be defined in this way. Suppose the chords are numbered arbitrarily, and the circle is traversed in an anti-clockwise sense starting from one end of chord 1. Let $i < j$. If chords i and j intersect but an end of chord j appears before any end of i as the circle is traversed, then the ij entry of I_γ is -1 and the ji entry is $+1$. Otherwise, I_γ is defined as above. Therefore I_γ need not have all of its negative entries above the main diagonal. However, I_γ is still skew-symmetric and $\det(I + I_\gamma)$ is unchanged and therefore equal to $Eu(D)$.

In the following sections we shall see some possible ways of generalising Theorem 1 and also some alternative ways of looking at the problem in an attempt to understand better why this theorem works. We shall also hint at possible connections with other areas of combinatorics.

3 A slight generalisation

Suppose we are given disjoint transpositions $\sigma_1 \ldots \sigma_t$ and we would like to write down all those products of transpositions which, when pre-multiplied with the cycle τ, still give a single cycle. Let J_γ be the matrix KI_γ where K is the diagonal matrix whose entry K_{ii} is equal to σ_i, where the σ_i are considered to be commuting variables. Then, using methods similar to the ones discussed above it can be shown that required products are given as a formal sum by exvaluating $\det(I + J_\gamma)$. An example of the use of this formula is given by Figure 5.

The sum of those products of transpositions from Figure 5 which, when pre-multiplied with τ, give a cyclic permutation is given by

$$\det \begin{pmatrix} 1 & \sigma_1 & \sigma_1 & \sigma_1 \\ -\sigma_2 & 1 & \sigma_2 & 0 \\ -\sigma_3 & -\sigma_3 & 1 & 0 \\ -\sigma_4 & 0 & 0 & 1 \end{pmatrix}$$

which equals

$$1 + \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 + \sigma_1\sigma_4 + \sigma_1\sigma_2\sigma_3\sigma_4.$$

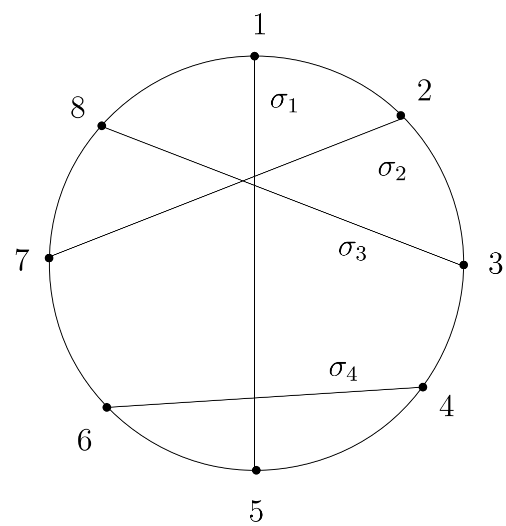


Figure 5: τ and some transpositions

It would be interesting to try using the results of Beck [1] in order to generalise the above to non-disjoint (and hence non-commuting) transpositions. Certainly more care needs to be taken here when defining the matrix J_γ.

4 A recurrence relation

The enumeration of Euler trails in D can be expressed in terms of a recurrence relation by using the idea of alternative routes. As always, suppose that the Euler trail γ is given. Let $1 \in V(D)$. Suppose the Euler trail γ goes through the arcs of 1 in the order $\ldots ab \ldots cd \ldots$. Let the arcs a, b, c, d be, respectively, $(a', 1), (1, b'), (c', 1)$ and $(1, d')$. Let D_1 be obtained from D by removing vertex 1 and adding the arcs (a', b') and (c', d'). Let D_1' be the digraph obtained from D by removing vertex 1 and this time adding the arcs (a', d') and (c', b') (the "alternative" route).

Then, since any Euler trail must take one of the two routes through vertex 1, it follows that

$$Eu(D) = Eu(D_1) + Eu(D_{1'})$$

with the conditions that $Eu(\text{simple cycle}) = 1$ and $Eu(\text{disconnected graph}) = 0$.

Figure 6 shows an example illustrating a step of this recurrence relation.

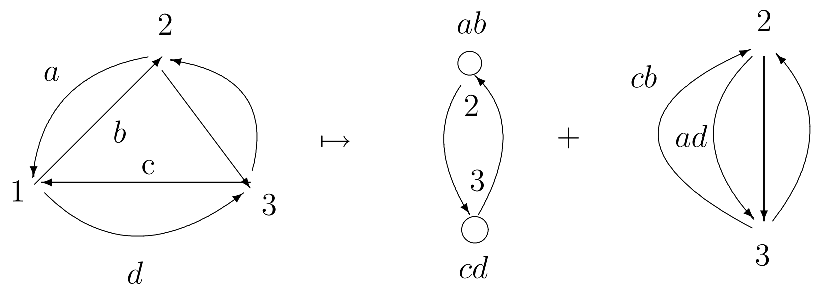


Figure 6: A step in the recurrence relation for calculating $Eu(D)$

The natural question here is to ask how this ties up (combinatorially) with the determinant formula $\det(I + I_\gamma)$ and whether or not this formula can be obtained as a result of the recurrence relation.

Moreover, if we are interested in $\det(I + J_\gamma)$ which, as discussed above, gives the sum of all those products of the σ's which give a cycle when pre-multiplied with τ, then this can be obtained by means of the recurrence relation

$$Eu(D) = Eu(D_1) + \sigma_1 Eu(D_{1'})$$

where here care has to be taken so that, in both D_1 and D_1', the order of the arcs is still that induced by the order in the Euler trail γ (this order gives an Euler trail in D_1 but not in D_1').

The above recurrence relations are also reminiscent of transformations carried out on a knot in order to calculate one of its polynomials, and one could ask if there is any connection between the above and with ideas from knot theory? In, particular, if all the σ_i were put equal to t in the second recurrence relation above, then a polynomial in t would arise. This polynomial would be equal to $\det(I + T_\gamma)$ where T_γ is the matrix obtained from I_γ by replacing every 1 and every -1 by t and $-t$, respectively.

Let this polynomial be $P_{D,\gamma}(t)$. This polynomial depends on both D and γ not just on D. For example, consider Figure 7 which shows a digraph D and two Euler trails γ_1 and γ_2.

Then

$$P_{D,\gamma_1}(t) = \det \begin{pmatrix} 1 & t & t & t \\ -t & 1 & 0 & t \\ -t & 0 & 1 & t \\ -t & -t & -t & 1 \end{pmatrix} = 1 + 5t^2$$

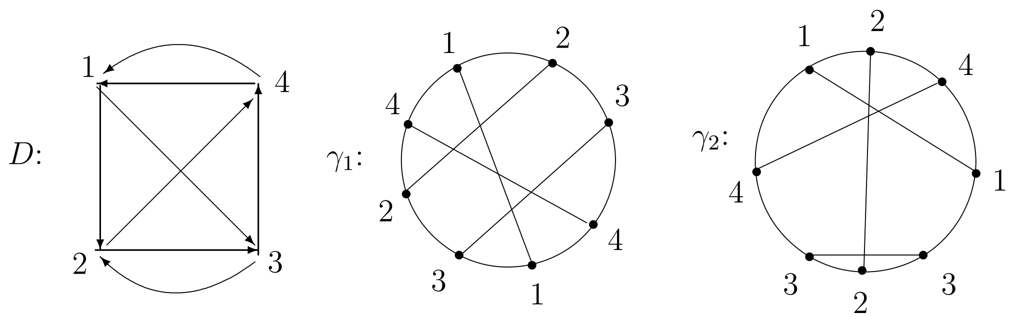


Figure 7: A digraph with two Euler trails

and

$$\det(I + J_{\gamma_1}) = 1 + \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2\sigma_4 + \sigma_3\sigma_4.$$

whereas

$$P_{D,\gamma_2}(t) = \det \begin{pmatrix} 1 & t & t & 0 \\ -t & 1 & t & t \\ -t & -t & 1 & 0 \\ 0 & -t & 0 & 1 \end{pmatrix} = 1 + 4t^2 + t^4$$

and

$$\det(I + J_{\gamma_2}) == 1 + \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 + \sigma_2\sigma_4 + \sigma_1\sigma_2\sigma_3\sigma_4.$$

Are there any relationships between different polynomials for different Euler trails γ for the same digraph D, except for the obvious fact that $P_{D,\gamma}(1)$ always equals $Eu(D)$?

5 The associated intersection digraph

Given the matrix $I + I_\gamma$ or, equivalently, the system of intersecting chords from which I_γ is calculated it is very natural to associate an *intersecting digraph* $G_{D,\gamma}$ whose vertices are the chords and such that there is an arc going from chord i to chord j if there is a $+1$ in the ij entry of I_γ. Also, $G_{D,\gamma}$ will have a loop at each vertex corresponding to the diagonal entries in $I + I_\gamma$. Traversing a loop or an arc in $G_{D,\gamma}$ along its direction will be associated with the weight $+1$ whereas traversing an arc against its direction will be associated with the weight -1.

For example, for the Euler trails γ_1, γ_2 of the digraph in Figure 7, the associated intersection digraphs are shown in Figure 9.

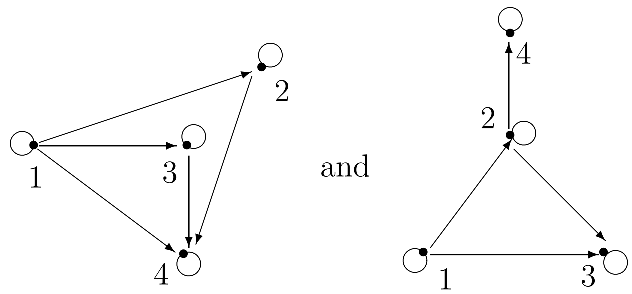


Figure 8: The intersection digraphs for the two Euler trails of Figure 7

Now it is known that the determinant of $I + I_\gamma$ be interpreted in terms of sums of products of weights of arcs of $G_{D,\gamma}$ taken around vertex-disjoint cycles of $G_{D,\gamma}$ which span it. What is the relationship between the structures of the intersection digraph $G_{D,\gamma}$ and D, and what is the relationship between the determinant of $I+I_\gamma$ interpreted as spanning cycles of $G_{D,\gamma}$, and the Euler trails in D?

6 Representing the intersection of chords by a quadratic form

An alternative way to the intersection matrix I_γ of representing the intersection of chords is to use a quadratic form as we shall now describe. Let chord i be represented by the variable s_i (we do not use $\sigma' s$ here in order to avoid confusion between the products presented in this section and the products of the $\sigma' s$ as described above which denote composition of transpositions). Then, if chords i and j intersect, this is represented by the product $s_i s_j$.

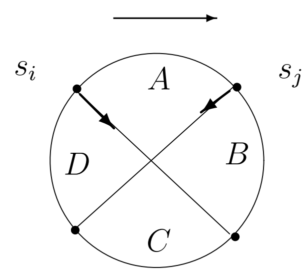
For example, for γ_1 shown in Figure 7, the corresponding quadratic form would be

$$s_1 s_2 + s_1 s_3 + s_1 s_4 + s_2 s_4 + s_3 s_4.$$

If alternative routes are taken at both vertices 1 and 2 we get γ_2 also shown in Figure 7 and which would be represented by the form

$$s_1 s_2 + s_1 s_4 + s_2 s_3.$$

Each of these forms would correspond to an Euler trail. Rules for transforming from one form to another could be formulated. How does the number



gives the term $s_i s_j$

Basic transformation:

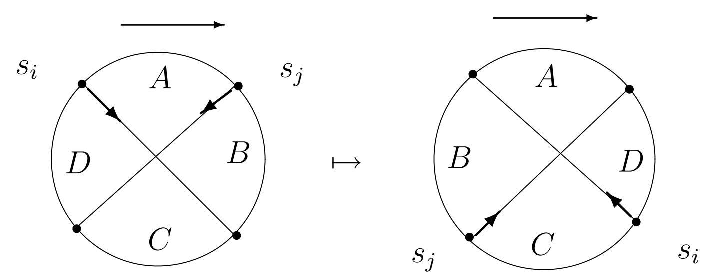


Figure 9: The rule for determining the coefficient for intersecting chords and the basic transformation

of such forms obtained according to these rules correspond to the determinant formula?

Actually some care is needed to ensure well-definition of these transformation rules (recall the modified definition of I_γ given above in order to cater for the case when the chords are not numbered in order, something which can happen after effecting a transformation corresponding to taking alternative routes at some vertices). More precisely, each form will have coefficients from the field $\{0, 1, -1\}$. Also, the variables s_i will be taken to satisfy $s_i^2 = 0$ and $s_i s_j = -s_j s_i$. Distributive laws are assumed to hold. Each chord is given an arbitrary direction. Figure 9 determines whether $s_i s_j$ will have a coefficient equal to $+1$ or -1.

A basic transformation will corresponding to taking the alternative route at two vertices whose chords intersect. The basic transformation, in terms of the quadratic form representation, then corresponds to

$$\begin{aligned} &s_i s_j + s_i p + s_j q + (s_i + s_j) r + Q \mapsto \\ &s_i s_j + s_i q + s_j p + (s_i + s_j) r + Q + qp + rp + pr \end{aligned}$$

where p is the sum of those chords (some coefficients could be -1) which

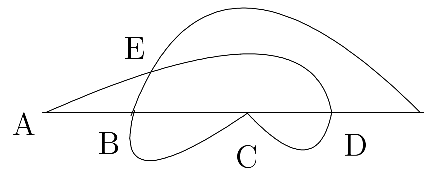


Figure 10: A curve with code $ABCDEBCDEA$

intersect s_i but not s_j; q is the sum of those chords which intersect s_j but not s_i; r is the sum of those chords which intersect both s_i and s_j; Q is the quadratic form containing intersections not involving either of the terms s_i or s_j.

Any two forms are equivalent if one can be obtained from the other by a sequence of basic transformations. How many forms equivalent to a given form are there? Is this equal to $Eu(D)$? What is the connection with $\det(I + I_\gamma)$?

7 A problem of Gauss

The following problem of Gauss could be related to the above discussion. Suppose a continuous closed curve γ with a finite number of self-intersections is drawn in the plane. The curve is also allowed to touch itself (this actually gives a slight generalisation of the original problem posed by Gauss but which seems more natural in the context of 2-in-2-out digraphs). Let the points of crossing or touching be labelled and write down the labels in the order they are encountered as the curve is traversed. This sequence is called the "code" of the curve, and an example is given in Figure 10.

The question posed by Gauss is to characterise those sequences which can arise as codes of a curve?

This question has been considered by a number of authors (for example, [4, 7, 9]). In a number of these works, ideas similar to those discussed above have been used (for example, "alternative routes" and the associated intersection graph). This is not so surprising since, after all, a curve in the Gauss problem, together with an orientation induced on it by traversing it, corresponds to a planar 2-in-2-out digraph.

Now, if the letters of a code of a curve are drawn on a circle and each pair of repeated letters is joined by a chord and the corresponding intersection matrix I_γ is written, then, by Theorem 3, the determinant of I_γ and any of its principal submatrices is 0 or 1.

Could this necessary condition form part of a sufficient condition for a sequence of letters (each letter appearing exactly twice) to be the code of a curve in the sense of Gauss?

8 Generalising to arbitrary digraphs

Finally, perhaps the most natural question is to ask for a generalisation of Theorem 1 to Eulerian digraphs with arbitrary degrees. Thus, let D be an Eulerian digraph with vertices $\{1, 2, \ldots, n\}$ and let the indegree and outdegree of vertex i be d_i. Let γ be an Eulerian trail in D and draw its representation as a circle with chords numbered consecutively (vertex i will appear d_i times round the circle). Given two vertices i, j erase the other vertices from the circle, giving a digraph D_{ij} with only two vertices i and j. Let e_{ij} be the number of Euler trails in D_{ij} (in the 2-in-2-out case, e_{ij} could only be 2 (if chords i, j intersected) or 1 (if they did not)).

Now if D had just two vertices, then it is clear that $Eu(D)$ would be equal to the determinant of the matrix

$$\begin{pmatrix} (d_1-1)! & \sqrt{e_{12}-(d_1-1)!(d_2-1)!} \\ -\sqrt{e_{12}-(d_1-1)!(d_2-1)!} & (d_2-1)! \end{pmatrix}$$

and that this matrix is equal to $I + I_\gamma$ when $d_1 = d_2 = 2$.

Therefore let us try defining the intersection matrix R_γ to be:

$$\begin{aligned} (R_\gamma)_{ii} &= (d_1-1)! \\ (R_\gamma)_{ij} &= \sqrt{e_{ij}-(d_i-1)!(d_j-1)!} \quad (i<j) \\ (R_\gamma)_{ij} &= -(R_\gamma)_{ij} \quad (i>j) \end{aligned}$$

Thus R_γ reduces to $I + I_\gamma$ if all the d_i equal 2 and it also gives the correct number of Euler trails for arbitrary d_i but $n = 2$.

However, $Eu(D)$ is not equal to $\det(R_\gamma)$! For example, consider the digraph shown in Figure 1.There we saw using the BEST formula that this digraph has 12 Euler trails. One possible Euler trail γ, represented as usual on a circle, is shown in Figure 11.

The above definition of R_γ would give the matrix

$$\begin{pmatrix} 2! & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 1 & 0 & 0 \\ -\sqrt{2} & 0 & 1 & 0 \\ -\sqrt{2} & 0 & 0 & 1 \end{pmatrix}$$

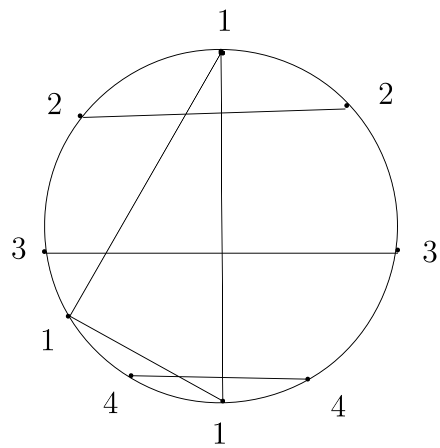


Figure 11: An Euler trail for the digraph in Figure 1

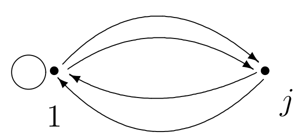


Figure 12: The digraph D_{1j}

Note that here, the graph D_{1j} is as shown in Figure 12; therefore $e_{1j} = 4$. Therefore $\sqrt{e_{ij} - (d_i - 1)!(d_j - 1)!} = \sqrt{4 - 2!1!} = \sqrt{2}$.

However this matrix has determinant equal to 8 not 12. Therefore Theorem 1 is not generalised by the above definition of R_γ and this question remains open.

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