Ordered Groups

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1 Ordered Monoids

An **ordered monoid** is a set with a monoid operation $\cdot$ and an order relation $\leqslant$, such that the operation is monotone:

$$x \leqslant y \;\Rightarrow\; ax \leqslant ay,\; xa \leqslant ya$$

Hence if $x \leqslant y$, $a \leqslant b$, then $ax \leqslant by$.

The morphisms are the monotone group morphisms (preserve both $\cdot$ and $\leqslant$). (*Left-ordered* monoids only have left multiplication being monotonic.)

Examples:

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	$0 <$	$2 <$	1
0	0	0	0
2	0	2	2
1	0	2	1

	a	b
a	a	ab
b	ab	b^2

$ab^2 = 0$
$0 \leqslant x \leqslant 1$

	a	b
a	a	ab
b	ab	b

$a < 1 < b$

	a	b
a	a	ab
b	ba	b

$1 \leqslant x$
$ab < ba$

Other finite examples:

- $0 = a^n < a^{n-1} < \cdots < a < 1$
- $0 < a < 1 < \top$, $a < b < \top$ with $a^2 = 0 = ab$, $b^2 = a = ba$
- $0 < a < b < c < \top$, $a < 1 < \top$, $xy = 0$ except $c^2 = a$, $x1 = x = 1x$, $x\top = x = \top x$.

• $\mathbb{Z}$ with addition and $\leqslant$.
  $\mathbb{N}^\times$ with multiplication and $\leqslant$ (but not $\mathbb{Z}^\times$ since $(-1)(-1) \not\leqslant (-1)1$).

• The endomorphism monoid of an ordered space with $\phi \leqslant \psi \Leftrightarrow \phi(x) \leqslant \psi(x), \forall x$. Every ordered monoid is embedded in some such space (e.g. via $x \mapsto f_x$, where $f_x(y) := xy$).

• *Free monoid*: Words with the operation of concatenation and $u \leqslant v$ if letters of u are in v in the same order, e.g. `abc`$\leqslant$`xaxxbxcx`. $1 \leqslant X$.

- *Divisibility monoids*: Any pure monoid modulo a normal subgroup of invertibles, X/G, with $xG \leqslant yG \Leftrightarrow x|y$, meaning $ax = y$ and $xb = y$ for some a, b. For example, $\mathbb{N}$ with $+$ and $\leqslant$; any $\vee$-semi-lattice with multiplication $\vee$ (so $y = a \vee x \Leftrightarrow y \geqslant x$); any integral domain with a field of fractions F and invertibles G induce the abelian ordered group $F^\times/G$. Satisfies $1 \leqslant X$.

  More generally, any cancellative monoid with a sub-monoid P which is central and whose only invertible is 1; let $x \leqslant y \Leftrightarrow y = xa, \exists a \in P$. For example, X^Y where X is a commutative ordered group and P is the set of monotonic functions which fix 1; $\mathbb{F}[x]$ with P the monic polynomials. Or any monoid with P the sub-monoid of central idempotents.

- *Generalized Minkowski space*: $\mathbb{R}^n$ with $\boldsymbol{x} \leqslant \boldsymbol{y} \Leftrightarrow \boldsymbol{y} - \boldsymbol{x} \in P$ where P is (a) $\mathbb{N} \times \boldsymbol{0}^{n-1}$, or in general (b) any sub-monoid generated from $A \subseteq \mathbb{R}^+ \times \mathbb{R}^{n-1}$ such as any convex rayed subset (e.g. cones).

- Any monoid with a zero and the inequalities $0 \leqslant x$.

Sub-monoids, products, and X^A are also ordered monoids. $X \times 1$ and $1 \times X$ are convex sub-monoids of $X \times Y$.

An ordered monoid can act on an ordered set, in which case $a \leqslant b$, $x \leqslant y$ implies $a \cdot x \leqslant b \cdot y$. If a monoid X acts on another Y, then their semi-direct (or ordinal) product is $X \rtimes Y$ with $(a, b)(x, y) := (a(a \cdot y), by)$ and the product or lexicographic order. In particular, the lex product $X \overleftarrow{\times} Y$ with $(a, b)(x, y) := (ax, by)$.

Since the intersection of convex sub-monoids is again of the same type, a subset A *generates* a unique smallest convex sub-monoid Convex(A). Any morphism pulls convex normal subgroups N to convex normal subgroups $\phi^{-1}N$. The map $x \mapsto a^{-1}xa$ is an automorphism.

Proposition 1

The completion of an ordered monoid is again an ordered monoid.

PROOF: Recall the Dedekind-MacNeille completion, where $A^{LU} := LU(A)$ and $x^{LU} = \downarrow x$. It is easy to prove $A^{LU}x = (Ax)^{LU}$, so $A^{LU}B \subseteq (AB)^{LU}$ and $(A^{LU}B)^{LU} = (AB)^{LU}$. On the completion $\bar{X}$ consisting of the 'closed' subsets $A^{LU} = A$, define $A \cdot B := (AB)^{LU}$. Then $(A \cdot B) \cdot C = ((AB)^{LU}C)^{LU} = (ABC)^{LU} = A \cdot (B \cdot C)$. The identity of $\bar{X}$ is $1^{LU} = \downarrow 1$ since $A \cdot 1^{LU} = (A1)^{LU} = A$. $A \subseteq B \Rightarrow (AC)^{LU} \subseteq (BC)^{LU}$ is trivial. X is embedded in $\bar{X}$ since $(xy)^{LU} = x^{LU}y^{LU}$.

□

1.0.1 Positive Cone

For any idempotent e, the subset $\uparrow e$ is an upper-closed directed sub-monoid ($x, y \geqslant e \;\Rightarrow\; xy \geqslant x, y$)

In particular, the *positive cone* of X is $X^+ := \uparrow 1 = \{\, x : x \geqslant 1 \,\}$; it is a convex normal sub-monoid ($a^{-1}xa \geqslant 1$). Similarly $X^- := \downarrow 1 = \{\, x : x \leqslant 1 \,\}$.

1. $a > 1$ AND $b \geqslant 1 \;\Rightarrow\; ab > 1$; $a, b \geqslant 1$ AND $ab = 1 \;\Rightarrow\; a = 1 = b$.
2. Any element of X^+ or X^- is either aperiodic or has period 1.

   Proof: $x^n \leqslant x^{n+1} \leqslant \ldots \leqslant x^{n+m} = x^n$.
3. The sub-monoid generated from $X^+ \cup X^-$ is connected.

   Proof: $x = a_+ b_- c_+ \cdots \geqslant b_- c_+ \cdots \leqslant c_+ \cdots \leqslant 1$.
4. If $\phi : X \to Y$ is a morphism then $\phi X^+ \subseteq Y^+$. For a sub-monoid $Y^+ = X^+ \cap Y$.
5. If $x_i y_j \leqslant y_j x_i$ then $x_1 \cdots x_n y_1 \cdots y_m \leqslant y_1 \cdots y_m x_1 \cdots x_n$. In particular, if $xy \leqslant yx$ then $x^n y^n \leqslant (xy)^n \leqslant (yx)^n \leqslant y^n x^n$.
6. A top $\top$ or bottom $\bot$ of the space are idempotents, but need not be the same as any of 1 and 0. However, if $0 < 1$ or $1 < 0$ holds, then 0 is the bottom or top (by duality, one can assume 0 to be the bottom).
7. If $0 < 1$ then X^+ has no zero divisors; dual statements hold.
8. The relation $x \prec y \Leftrightarrow \exists n \in \mathbb{N}^+,\ x \leqslant y^n$ is a pre-order relation on X^+; it induces an equivalence relation $x \prec y$ AND $y \prec x$ with equivalence classes called *Archimedean components*; 1 is its own equivalence class; one can define $[x] \prec [y]$ when $x \prec y$. Note that $x \prec y \;\Rightarrow\; \phi(x) \prec \phi(y)$, $x^n \in [x]$, $x, y \prec a \;\Rightarrow\; xy \prec a$. One also writes $x \ll y$ for $x \prec y$ but $y \not\prec x$, meaning x is "infinitesimal" compared to y.

   X is called "isolating" when $1 \prec y \Leftrightarrow 1 \leqslant y$.
9. When X is commutative, each Archimedean component together with 1 is a sub-monoid. An *Archimedean monoid* is the case when there is only one non-trivial component, so

   $$1 < x \leqslant y \;\Rightarrow\; \exists n \in \mathbb{N},\ y < x^n$$

   i.e., $x^{\mathbb{N}}$ is unbounded for $x > 1$.

1.1 The Group $\mathcal{G}(X)$ of Invertibles

1. $\mathcal{G}(X)$ is either trivial $\{\, 1 \,\}$ or it has no maximum and minimum.

   Proof: If $a \geqslant 1$ is a maximum then $a \leqslant a^2 \leqslant a$, so $a = 1$.

2. If $a > 1$ is invertible, then it is aperiodic $\cdots < a^{-1} < 1 < a < a^2 < \cdots$. Periodic invertibles are incomparable to 1; so $\mathcal{G}^{+/-}$ are torsion-free: $x^n = 1 \Leftrightarrow x = 1$ $(n \geqslant 1)$.

   If a is invertible then $\uparrow a = aX^+ = X^+a$ is order-isomorphic to X^+ (via $x \mapsto a^{-1}x$).

   Thus finite ordered groups have trivial order.

3. The order structure of $\mathcal{G}$ is determined by $\mathcal{G}^+$, $x \leqslant y \Leftrightarrow x^{-1}y \in \mathcal{G}^+$. $\mathcal{G}^+$ and $\mathcal{G}^-$ are closed under multiplication and conjugation. (Hence $\mathcal{G}X^+ = X^+\mathcal{G}$ and $\mathcal{G}X^-$ are sub-monoids.)

   (For any group, one can pick any sub-monoid for $\mathcal{G}^+$ with the property that if $x \in \mathcal{G}^+$, $x \neq 1$, then $x^{-1} \notin \mathcal{G}^+$, and define $x \leqslant y \Leftrightarrow ax = y, xb = y$ for some $a, b \in \mathcal{G}^+$.)

4. $\mathcal{G}^-$ is a mirror image of $\mathcal{G}^+$ via the quasi-complement map $x \mapsto x^{-1}$,

   $$x \leqslant y \Leftrightarrow y^{-1} \leqslant x^{-1}$$

   so $x \in \mathcal{G}^+ \Leftrightarrow x^{-1} \in \mathcal{G}^-$; $\mathcal{G}^+ \cap \mathcal{G}^- = \{1\}$.

5. A subgroup Y is convex $\Leftrightarrow$ Y^+ is convex in X^+.

   The kernel of an ordered-group morphism $\phi : G \to H$ is a convex normal subgroup. Conversely, for Y a convex normal subgroup, G/Y is a left-ordered group, with

   $$gY.hY := (gh)Y, \qquad gY \leqslant hY \Leftrightarrow gy_1 \leqslant hy_2,\ \exists y_1, y_2 \in Y$$

   (anti-symmetry requires convexity); then $G/\ker\phi \cong \operatorname{im}\phi$.

6. $[x_i, y_j] > 1 \Rightarrow [x_1 \cdots x_n, y_1 \cdots y_m] > 1$ (since $[x, ab] = [x, b]b^{-1}[x, a]b$).

7. (Rhemtulla) The ordered group G is determined by its group ring $\mathbb{Z}G$ (which can be embedded in a division ring).

1.2 Residuated Monoids

are ordered monoids such that for every pair x, y, there are elements $x \to y$ and $x \leftarrow y$,

$$xw \leqslant y \Leftrightarrow w \leqslant (x \to y)$$
$$wx \leqslant y \Leftrightarrow w \leqslant (y \leftarrow x)$$

equivalently the maps $x*$ and $*x$ have adjoints $x\to$ and $\leftarrow x$; equivalently $x \to y$ is the largest element such that $x(x \to y) \leqslant y$, and similarly $(y \leftarrow x)x \leqslant y$.

(Dual relations: $xw \geqslant y \Leftrightarrow w \geqslant (x \backslash y)$, etc.)

Examples:

- Ordered groups, with $x \to y = x^{-1}y$, $y \leftarrow x = yx^{-1}$, $x^{-1} = x \to 1$. A residuated monoid is a group when $x(x \to 1) = 1 = (x \to 1)x$.

- The subsets of any monoid with $AB := \{ ab : a \in A, b \in B \}$ and $A \subseteq B$; then $A \to B = \{ x : Ax \subseteq B \}$, $B \leftarrow A = \{ x : xA \subseteq B \}$. It has a zero $\varnothing$ and an identity $\{ 1 \}$ (the order is Boolean but it need not be a lattice monoid).
- The additive subgroups of a unital ring with $A * B := [\![AB]\!] = \{ \sum_{i=1}^{n} a_i b_i : a_i \in A, b_i \in B \}$ and $A \subseteq B$; has a zero 0, an identity $[\![1]\!]$, is modular; $A \to B = \{ x : Ax \subseteq B \}$.
- *Bicyclic Monoid* $[\![a, b : ba = 1]\!]$ with free monoid order; then $a^m b^n \leqslant a^{m+r} b^{n+r}$, idempotents are $a^n b^n$. Equivalently, $\mathbb{N}^2$ with $(m, n)(i, j) := (m - n + \max(n, i), j - i + \max(n, i))$.

In what follows, every inequality has a dual form in which every occurrence of $x \to y$ and xy are replaced by $y \leftarrow x$ and yx.

1. By the general results of adjoints, $x*$ and $*x$ preserve $\leqslant$, and
$$x(x \to y) \leqslant y \leqslant x \to (xy), \qquad x(x \to xy) = xy,$$
$$x \to x(x \to y) = x \to y$$
$$y \leqslant z \;\Rightarrow\; x \to y \leqslant x \to z$$
$$y \leqslant z \;\Rightarrow\; y \to x \geqslant z \to x$$
Proof: If $y \leqslant z$ then $w \leqslant (x \to y) \Leftrightarrow xw \leqslant y \Rightarrow xw \leqslant z \Leftrightarrow w \leqslant (x \to z)$.
2. $1 \to x = x = x \leftarrow 1$, $x \to x \geqslant 1$, $x(x \to x) = x$.
3. $(z \to y)x \leqslant (z \to yx)$, $x \to y \leqslant zx \to zy$, $(x \to 1)y \leqslant x \to y$.
(since $z(z \to y)x \leqslant yx$)
4. (a) $x \to (y \to z) = (yx) \to z$, hence $x \to y \leqslant (z \to x) \to (z \to y)$
(b) $x \to y \leftarrow z$ is unambiguous.
(c) $x \leqslant y \leftarrow (x \to y)$
Proof: $w \leqslant x \to (y \leftarrow z) \Leftrightarrow xw \leqslant y \leftarrow z \Leftrightarrow xwz \leqslant y \Leftrightarrow w \leqslant (x \to y) \leftarrow z$
5. $(x \to y)(y \to z) \leqslant (x \to z)$, $(x \to x)(x \to x) = x \to x$
Hence $x \to y \leqslant (x \to z) \leftarrow (y \to z)$
6. If a bottom 0 exists, then it is a zero $x0 = 0 = 0x$; there would also be a top $\top = 0 \to 0 = 0 \leftarrow 0$, so $0 \to x = \top = x \to \top$. $x \to 0 \neq 0$ iff x is a divisor of zero.
7. When 1 is the top of the order, $\leftarrow, \to$ are implications,
$$x \leqslant y \Leftrightarrow x \to y = 1$$
in particular $(x \to 1) = 1 = (x \to x) = (0 \to x)$.
8. When $*$ is commutative, $x \to y = y \leftarrow x$.

2 Lattice Monoids

are sets with a monoid operation and a lattice order such that multiplication is a lattice morphism,

$$x(y \vee z) = (xy) \vee (xz) \qquad x(y \wedge z) = (xy) \wedge (xz)$$
$$(y \vee z)x = (yx) \vee (zx) \qquad (y \wedge z)x = (yx) \wedge (zx)$$

They are ordered monoids since $x \leqslant y \Leftrightarrow x \vee y = y \Rightarrow ax \vee ay = ay \Leftrightarrow ax \leqslant ay$. But, conversely, an ordered monoid whose order is a lattice can only guarantee $x(y \vee z) \geqslant (xy) \vee (xz)$, etc.

Examples:

- The endomorphisms of a lattice with composition and
$$(\phi \vee \psi)(x) = \phi(x) \vee \psi(x).$$
- Any distributive lattice with $\wedge$ as the operation.
- Free monoids of words from a finite alphabet with operation of joining and linearly ordered according to first how many `a`'s, then `b`, `ab`, `ba`, `aab`, etc.,
$$\_ < \mathtt{b} < \mathtt{bb} < \cdots < \mathtt{a} < \mathtt{ba} < \mathtt{ab} < \mathtt{bba} < \mathtt{bab} < \mathtt{abb} < \mathtt{aa} <$$
$$\mathtt{baa} < \mathtt{aba} < \mathtt{aab} < \mathtt{bbaa} < \mathtt{baba} < \mathtt{abba} < \mathtt{baab} < \cdots$$
Equivalently, replace a by $(1+a)$, etc., expand the resulting polynomials, and compare using first degrees then lexicographic (for same degree).
- Factorial monoids (i.e., those that have unique factorizations into irreducibles) with $x \leqslant y \Leftrightarrow x|y$, e.g. $\mathbb{Q}[x]$.

A lattice-sub-monoid is a subset that is closed under 1, $*$, $\wedge$, $\vee$. $X \times Y$ and X^A are lattice monoids. Morphisms need to preserve both the monoid and lattice structure.

1. (a) $(x \vee y)(a \vee b) = (xa) \vee (ya) \vee (xb) \vee (yb)$,
   (b) $(x \vee y)(a \wedge b) = (xa \wedge xb) \vee (ya \wedge yb) = (xa \vee ya) \wedge (xb \vee yb)$.
   (c) $xa \wedge yb \leqslant (x \vee y)(a \wedge b) \leqslant xa \vee yb$
   (d) If x, y commute then $xy = (x \vee y)(x \wedge y)$, and
   $$(x \vee y)^n = x^n \vee x^{n-1}y \vee \cdots \vee y^n, \quad (x \wedge y)^n = x^n \wedge \cdots \wedge y^n.$$
   Note, in general, $x \vee (yz) \neq (x \vee y)(x \vee z)$.
2. X^+, X^- are sub-lattice-monoids that generate X:
   Let $x_+ := x \vee 1$, $x_- := x \wedge 1$;

(a) $x_- \leqslant x \leqslant x_+$ with $x_\pm \in X^\pm$.

(b) $x = x_+ x_- = x_- x_+$.

(c) $x \mapsto x_+$ is a $\vee$-morphism and a closure map

$$x \leqslant y \;\Rightarrow\; x_+ \leqslant y_+, \qquad (x \vee y)_+ = x_+ \vee y_+, \qquad (x \wedge y)_+ \leqslant x_+ \wedge y_+$$

Dually, $x \mapsto x_-$ is a $\wedge$-morphism,

$$x \leqslant y \;\Rightarrow\; x_- \leqslant y_-, \qquad (x \wedge y)_- = x_- \wedge y_-, \qquad x_- \vee y_- \leqslant (x \vee y)_-$$

$$x_{++} = x_+, \quad x_{+-} = 1 = x_{-+}, \quad x_{--} = x_-$$

(d) $x_- y_- \leqslant x_- \wedge y_- \leqslant x \wedge y \leqslant x_+ y_- \leqslant x \vee y \leqslant x_+ \vee y_+ \leqslant x_+ y_+$
$x_- y_- \leqslant (xy)_- \leqslant (x_+ y)_- \leqslant x_+ y_- \leqslant (xy_-)_+ \leqslant (xy)_+ \leqslant x_+ y_+$

(e) If x, y commute, then so do $x_\pm, y_\pm$.

(f) Morphisms preserve $x_\pm$, e.g. $(a^{-1}xa)_\pm = a^{-1} x_\pm a$.

Proof: $(x \vee 1)(x \wedge 1) = x1$ by 1(d). $x_+ y_- = (1 \vee x)(1 \wedge y) = (1 \wedge y) \vee (x \wedge yx) = y_- x_+$.

3. $x^n \geqslant 1 \;\Leftrightarrow\; x \geqslant 1$, $x^n = 1 \;\Leftrightarrow\; x = 1$, $x^n \leqslant 1 \;\Leftrightarrow\; x \leqslant 1$.

   Proof: If $x^n \geqslant 1$ then $x_-^{n+1} = x_-(1 \wedge \cdots \wedge x^{n-1}) = x_-^n$ so $x^{n+1} = x^n x_+ \geqslant 1$. If $x^2 \geqslant 1$ then $x = x_+ x_- = (1 \vee x) \wedge (x \vee x^2) = (1 \vee x) \wedge (1 \vee x)^2 = x_+ \wedge x_+^2 \geqslant 1$; thus $x^{2^n} \geqslant 1 \;\Rightarrow\; x \geqslant 1$.

   So every invertible element, except 1, is aperiodic; its generated subgroup is isomorphic to $\mathbb{Z}$ as $\ldots < a^{-2} < a^{-1} < 1 < a < a^2 < \ldots$ or they are mutually incomparable.

4. $x^n a \leqslant a y^n \;\Leftrightarrow\; xb \leqslant by$ for some a, b.

   Proof: Let $b := x^{n-1} a \vee x^{n-2} a y \vee \cdots \vee a y^{n-1}$.

5. $\mathcal{G}(X)$ is a lattice subgroup, since for invertible elements,

$$(x \vee y)^{-1} = x^{-1} \wedge y^{-1}, \quad x \vee y = x(x \wedge y)^{-1} y,$$

$$(x^{-1})_+ = (x_-)^{-1}, \quad (x^{-1})_- = (x_+)^{-1}$$

$$x \vee x^{-1} \geqslant 1$$

   Proof: $1 \leqslant (x \vee y)(x^{-1} \wedge y^{-1}) \leqslant 1$. $(x \vee x^{-1})^2 = x^2 \vee 1 \vee x^{-2} \geqslant 1$.

6. In X^+, x and y are said to be *orthogonal* $x \perp y$ when $x \wedge y = 1$ and $xy = yx$. For $x \perp y$,

   (a) $(xy)_- = x_- y_-$

   (b) $1 \leqslant z \;\Rightarrow\; x \wedge (yz) = x \wedge z$

   (c) $x \perp z \;\Rightarrow\; x \perp (yz)$

(d) $x^n \perp y^m$ $(n, m \geqslant 1)$

(e) $1 \leqslant z \prec y \ \Rightarrow\ x \perp z$

Proof: $x \wedge yz = x(x \wedge y \wedge z) \wedge yz = (x \wedge y)(x \wedge z) = x \wedge z$.

Mutually orthogonal positive elements generate a free abelian group.

Proof: If $p \cdots = q \cdots$, then $1 = p \wedge (q \cdots) = p \wedge (p \cdots) = p$.

7. a is cancellative iff $ax \leqslant ay \ \Rightarrow\ x \leqslant y$.

8. The center $Z(X)$ is a sub-lattice-monoid.

9. An element in X^+ is called *irreducible* when for any $x, y \geqslant 1$,

$$a = xy \ \Rightarrow\ a = x \text{ OR } a = y.$$

In particular are the *primes*, when for any $x, y \geqslant 1$,

$$a \leqslant xy \ \Rightarrow\ a \leqslant x \text{ OR } a \leqslant y.$$

For example, atoms of X^+.

Proof: $x, y \leqslant xy = a \leqslant x$ or y. If $a \wedge x, a \wedge y < a$ then $a \wedge x = 1 = a \wedge y$, so $a \wedge xy = 1$.

2.1 Residuated Lattice Monoids

are residuated monoids which are lattice ordered. They are lattice monoids.

Examples:

- $\mathbb{N}$ with $m \to n = \text{quotient}(n/m)$.
- $[0, 1]$ with $xy := \max(0, x + y - 1)$; then $x \to y = \min(1, 1 - x + y)$.
- The set of relations on X with the operation of composition and $\cap, \cup$. Then $\rho \to \sigma = \{\, (x, y) : \rho x \subseteq \sigma y \,\}$ and $\rho \leftarrow \sigma = \{\, (x, y) : \rho^{-1} y \subseteq \sigma^{-1} x \,\}$.
- The ideals of a ring; the modules of a ring; complete lattice monoids. Much of the theory of ideals of rings generalizes to residuated lattice monoids.
- *Brouwerian algebra*: residuated lattice monoids in which $xy = x \wedge y$; they are commutative and distributive lattices with $X \leqslant 1$; a Heyting algebra is the special case of a bounded Brouwerian algebra, while a generalized Boolean algebra is the special case where $(x \to y) \to y = x \vee y$. Such examples can act as generalizations of classical logic.
- Matrices with coefficients from a Boolean algebra, with $A \leqslant B \Leftrightarrow \forall i, j,\ a_{ij} \leqslant b_{ij}$ and $AB = [\bigvee_k a_{ik} \wedge b_{kj}]$; then $A \wedge B = [a_{ij} \wedge b_{ij}]$, $A' = [a'_{ij}]$, $A \to B = (A^\top B')'$, $B \leftarrow A = (B' A^\top)'$.

1. $x(y \vee z) = (xy) \vee (xz)$; more generally, $(\bigvee A)(\bigvee B) = \bigvee_{a\in A, b\in B} ab$.

   Proof: $xy, xz \leqslant x(y \vee z)$; $xy, xz \leqslant xy \vee xz =: w$, so $y, z \leqslant x \to w$ and $x(y \vee z) \leqslant x(x \to w) \leqslant w$.

2. $x\to, \leftarrow x$ are $\wedge$-morphisms; $x\leftarrow, \to x$ are anti-$\vee$-morphisms,

$$x \to (y \wedge z) = (x \to y) \wedge (x \to z)$$
$$(y \vee z) \to x = (y \to x) \wedge (z \to x)$$

   Proof: $w \leqslant x \to (y \wedge z) \Leftrightarrow xw \leqslant y \wedge z \Leftrightarrow xw \leqslant y, z \Leftrightarrow w \leqslant x \to y$ AND $w \leqslant x \to z$.

   More generally, $(\bigvee A) \to x = \bigvee_{a\in A}(a \to x)$, $x \to (\bigwedge A) = \bigwedge_{a\in A}(x \to a)$.

3. X^- is again residuated with $x \to_- y = (x \to y)_-$, $x \leftarrow_- y = (x \leftarrow y)_-$.

4. Left/right conjugates of x by a are defined as $(a \to xa)_-$, $(ax \leftarrow a)_-$.

5. a is left cancellative iff $a \to ax = x$ (in particular $a \to a = 1$).

   Proof: $w \leqslant a \to ax \Leftrightarrow aw \leqslant ax \Leftrightarrow w \leqslant x$.

A *basic logic algebra* is a bounded residuated lattice monoid such that $x(x \to y) = x \wedge y = (x \leftarrow y)x$ and $(x \to y) \vee (y \to x) = 1$ (hence distributive and $X \leqslant 1$). A *GMV-algebra* is a bounded residuated lattice monoid such that $y \leftarrow x \to y = x \vee y$.

2.2 Lattice Monoids with $X^- \subseteq \mathcal{G}(X)$

Example: A residuated lattice monoid that satisfies $x(x \to y)_+ = x \vee y = (y \leftarrow x)_+ x$ (since if $x \leqslant 1$ then $x \to 1, 1 \leftarrow x \geqslant 1$, so $x(x \to 1) = x \vee 1 = 1 = (1 \leftarrow x)x$).

1. $x_+ \wedge (x_-)^{-1} = 1$; hence $(x_+)^n \perp (x_-)^{-m}$.

   Proof: If $y \leqslant x_+, (x_-)^{-1}$, then $x_- y \leqslant 1$ and $x_- y \leqslant x$, so $x_- y \leqslant x_-$.

2. The decomposition $x = x_+ x_-$ is the unique one such that $x_+ \in X^+$, $x_- \in X^-$, $x_+ \perp x_-^{-1}$.

   Proof: If $x = ab$, then $b = (a \wedge b^{-1})b = x_-$, so $a = x_+ x_- b^{-1} = x_+$.

3. The *absolute value* of an element is $|x| := x_+ x_-^{-1} = x_+ \vee x_-^{-1}$.

   (a) $1 \leqslant |x|$, $|x| = 1 \Leftrightarrow x = 1$,

   (b) $x \leqslant |x|$, $|x| = \begin{cases} x & \text{when } x \geqslant 1 \\ x^{-1} & \text{when } x \leqslant 1 \end{cases}$

   (c) $a \leqslant x \leqslant b \Rightarrow |x| \leqslant |a| \vee |b|$

   (d) $|xy| \leqslant x_+ |y| x_-^{-1}$; if x, y commute, then $|xy| \leqslant |x||y|$.

   (e) $|x \wedge y|, |x \vee y| \leqslant |x| \vee |y| \leqslant |x||y|$.

(f) If x, y are invertible, then

i. $|x| = x \vee x^{-1} = |x^{-1}|$,

ii. $|x|^{-1} = x \wedge x^{-1}$, so $|x|^{-1} \leqslant x \leqslant |x|$,

iii. $|xy| = (x \vee y^{-1})(x^{-1} \vee y)$.

(g) Morphisms preserve $|\cdot|$, $\phi(|x|) = |\phi(x)|$, in particular $|x^{-1}yx| = x^{-1}|y|x$.

Proof: If $x_+ \leqslant y$, $1 \leqslant x_- y$, then $x_+ \leqslant y \wedge xy = x_- y$. $a \leqslant x \leqslant b$ implies $x_+ \leqslant b_+$, $x_-^{-1} \leqslant a_-^{-1}$, so $|x| = x_+ \vee x_-^{-1} \leqslant |b| \vee |a|$. $|x \vee y| = (x \vee y)_+ \vee (x \vee y)_-^{-1} \leqslant x_+ \vee y_+ \vee (x_-^{-1} \wedge y_-^{-1}) \leqslant |x| \vee |y|$. For x invertible, $|x| = x_+ x_-^{-1} = (1 \vee x)(1 \wedge x)^{-1} = (1 \vee x)(1 \vee x^{-1}) = x \vee x^{-1} \geqslant 1$. $(x \vee y^{-1})(x^{-1} \vee y) = 1 \vee xy \vee (xy)^{-1} = 1 \vee |xy|$.

4. $(x^n)_+ = (x_+)^n$, $(x^n)_- = (x_-)^n$, $|x^n| = |x|^n$.

   Proof: $(x_-)^n = (x_+^n \wedge x_-^{-n})x_-^n = x^n \wedge 1 = (x^n)_-$; $x_+^n = x_+^n x_-^{-n} x_-^n = (x_+^n \vee x_-^{-n})x_-^n = x^n \vee 1$.

5. (Riesz Decomposition) For $a_i \in X^-$, $[a_1 \cdots a_n, 1] = [a_1, 1] \cdots [a_n, 1]$, i.e.,

$$ab \leqslant x \leqslant 1 \text{ AND } a, b \leqslant 1 \ \Rightarrow\ x = cd \text{ where } a \leqslant c \leqslant 1, b \leqslant d \leqslant 1$$

   Proof: Given $ab \leqslant x \leqslant 1$, $a, b \in X^-$, let $b := a \vee x$ and $d := xb^{-1} = x(x^{-1} \wedge a^{-1}) \geqslant 1 \wedge b = b$.

6. For $x_i, y_j \leqslant 1$, $\prod_{i,j}(x_i \vee y_j) \leqslant (x_1 \cdots x_n) \vee (y_1 \cdots y_m)$.

   Proof: It is enough to show $(x \vee y)(x \vee z) \leqslant x \vee yz =: s$; $yz \leqslant s \leqslant 1$, so $s = ab$ with $y \leqslant a \leqslant 1$, $z \leqslant b \leqslant 1$; so $x \leqslant ab \leqslant a$, hence $x \vee y \leqslant a$; similarly, $x \vee z \leqslant b$, and $(x \vee y)(x \vee z) \leqslant ab = s$.

7. If $a_i, b_j \leqslant 1$ and $a_1 \cdots a_n = b_1 \cdots b_m$, then there are unique $c_{ij} \leqslant 1$ such that $a_i = c_{i1} \cdots c_{im}$, $b_j = c_{1j} \cdots c_{nj}$, $c_{i+1,j} \cdots c_{n,j} \perp c_{i,j+1} \cdots c_{i,m}$.

   Proof: For $a_1 a_2 = b_1 b_2$, let $c_{11} := a_1 \vee b_1$, $c_{12} := c_{11}^{-1} a_1$, $c_{21} := c_{11}^{-1} b_1$, $c_{22} := a_1^{-1} c_{11} b_2 = a_2 \vee b_2$. Then $c_{21} c_{22} = c_{11}^{-1} b_1 (a_2 \vee b_2) = a_2$.

8. A sub-monoid is a convex lattice-sub-monoid when $|x| \leqslant |h| \ \Rightarrow\ x \in H$ for any $h \in H$. Its convex closure is thus

$$|H| := \{\, x : |x| \leqslant |h|, \exists h \in H \,\}.$$

   Proof: $|h_+| \leqslant |h|$, $|h_-^{-1}| \leqslant |h|$, and $|h \vee g| \leqslant |h||g| = ||h||g||$, so $h_\pm, |h|, h \vee g \in H$; if $h \leqslant x \leqslant g$ then $|x| \leqslant |h| \vee |g|$. $1 \leqslant x_+ \vee x_-^{-1} = |x| \leqslant h \in H$, so $x = x_+ x_- \in H$.

9. An *ultrametric* valuation is one which satisfies $|xy| \leqslant |x| \vee |y|$; so $|x^n| = |x|$.

2.3 Lattice Groups

are ordered groups whose order is a lattice. They are residuated, hence satisfy $x(y \vee z) = xy \vee xz$, but also $x(x \to y) = x$ and $x(x \to y)_+ = x \vee y$.

Examples:

- $\mathbb{Q}^\times$ with multiplication and $p \leqslant q \Leftrightarrow q/p \in \mathbb{N}$. It is Archimedean.
- The automorphism group of a lattice, e.g. $\mathbb{Z}$ with $+, \leqslant$; $\mathrm{Aut}_\leqslant(\mathbb{Q})$; $\mathrm{Aut}[0,1]$ is simple. Every lattice group is embedded in an automorphism group of some linear order.
- $C(X, Y)$ where Y is a lattice group; also measurable functions $X \to \mathbb{R}$.
- $X \rtimes_\phi Y$ is a lattice group if X is a lattice group and Y is a linearly ordered group.

Lattice groups are infinite, torsion-less, $\top$-less and $\bot$-less (except for the trivial group). (Strictly speaking, a lattice must have a top/bottom, but these cannot be invertible.) There is no equational property that characterizes lattice groups among groups, or among lattices.

1. A subgroup is a lattice when it is closed under $\vee$, or even just $x \mapsto x_+$, since $x \wedge y = (x^{-1} \vee y^{-1})^{-1}$, $x \vee y = x(x^{-1}y)_+$.

2. $x \mapsto ax$ is a $(\vee, *)$-automorphism, so the lattice is homogeneous.

   $\bigvee_i ax_i = a \bigvee_i x_i$ (since $ax_i \leqslant b \Leftrightarrow \bigvee_i x_i \leqslant a^{-1}b$).

3. The lattice is distributive, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
   Hence
   $$(x \vee y)_\pm = x_\pm \vee y_\pm, \qquad (x \wedge y)_\pm = x_\pm \wedge y_\pm,$$
   $$x_+ \wedge y = (x \wedge y) \vee y_-, \qquad x_- \vee y = (x \vee y) \wedge y_+$$

   Proof: $x \wedge (y \vee z) \leqslant (y \vee z)y^{-1}x \wedge (y \vee z) = (y \vee z)(y^{-1}x \wedge 1) = (y \vee z)y^{-1}(x \wedge y)$. Hence $(x \wedge (y \vee z))((x \wedge y)^{-1} \wedge (x \wedge z)^{-1}) \leqslant (y \vee z)(y^{-1} \wedge z^{-1}) = 1$, so $x \wedge (y \vee z) \leqslant (x \wedge y) \vee (x \wedge z)$.

   By the same argument, $x \wedge \bigvee_i x_i = \bigvee_i (x \wedge x_i)$ for complete lattice groups.

4. $x = ab$, where $b \leqslant 1 \leqslant a$, iff $a = x_+t$, $b = t^{-1}x_-$ (since $t := x_+^{-1}a = x_-b^{-1}$).

5. $x_i \wedge y_j \leqslant 1 \;\Rightarrow\; (x_1 \cdots x_n) \wedge (y_1 \cdots y_m) \leqslant 1$

   Proof: It is enough to show $x \wedge y \leqslant 1$, $x \wedge z \leqslant 1$ imply $x \wedge yz \leqslant 1$. Let $a := y \vee z$; then $(1 \vee ax)^{-1}(x \wedge a^2) = x \wedge x^{-1}a^{-1}x \wedge x^{-1}a \wedge a^2 \leqslant 1 \wedge a^2 \leqslant 1$ (using $s \wedge t \leqslant (st)_+$), so $x \wedge a^2 \leqslant 1 \vee ax$; so $x \wedge yz \leqslant x \wedge a^2 = (x \wedge a^2) \wedge (1 \vee ax) = (x \wedge a^2)_- \vee (x \wedge a(x \wedge a)) \leqslant 1$.

6. $(xy)_+ = x_+(x_- \vee y_+^{-1})(x_+^{-1} \vee y_-)y_+$.

   $|x \vee y| = (x \vee |y|) \wedge (|x| \vee y)$.

7. If x, y commute, then

   (a) $x^n \leqslant y^n \Rightarrow x \leqslant y$.

   (b) $(x \vee y)^n = x^n \vee y^n$, $(x \wedge y)^n = x^n \wedge y^n$.

   Proof: $(x \vee y)^n = (x(x^{-1}y)_+)^n = x^n(x^{-n}y^n)_+ = x^n \vee y^n$.

8. $x, y \in G^+$ are orthogonal when

$$x \wedge y = 1 \Leftrightarrow x \vee y = xy$$

   (since $xy = x(x \wedge y)^{-1}y = x \vee y$).

   More generally, for mutually orthogonal elements, $x_1 \cdots x_n = x_1 \vee \cdots \vee x_n$ (by induction, since $xy \wedge z = (x \vee y) \wedge z = 1$).

9. If $|x| \perp |y|$ then $yx = xy$, $(xy)_+ = x_+y_+$, $(xy)_- = x_-y_-$, $|xy| = |x||y| = |x| \vee |y|$.

   Proof: $1 \leqslant x_+ \wedge y_-^{-1} \leqslant |x| \wedge |y| = 1$, etc., so $x_\pm$, $y_\pm$ commute. $xy = x_+y_+x_-y_-$, but $(x_+y_+) \wedge (x_-y_-)^{-1} = (x_+ \vee y_+) \wedge (x_-^{-1} \vee y_-^{-1}) = 1$, so by uniqueness, $(xy)_+ = x_+y_+$, $(xy)_- = x_-y_-$; thus $|xy| = (xy)_+(xy)_-^{-1} = x_+y_+x_-^{-1}y_-^{-1} = |x||y|$.

10. (a) The $\vee$-irreducible elements of G^+ are those a such that $[1, a]$ is a chain.

    (b) The prime elements of G^+ are its atoms. They are mutually orthogonal and generate a free abelian normal convex lattice subgroup ($\cong \mathbb{Z}^{(A)}$).

    Proof: $a = x(x \vee y)^{-1}a \vee y(x \vee y)^{-1}a$, so $a = x(x \vee y)^{-1}a$, say, i.e., $y \leqslant x$. If $1 \leqslant x \leqslant a$ then $a = xx^{-1}a$, so $a \leqslant x$ or $a \leqslant x^{-1}a$, i.e., $x = a$ or $x = 1$.

11. A group morphism which preserves $\phi(x_+) = \phi(x)_+$, or equivalently orthogonality, is a morphism (since $\phi(x \vee y) = \phi(x)\phi(x^{-1}y)_+ = \phi(x) \vee \phi(y)$; $1 = x \wedge y = x(x^{-1}y)_+$, $x_+ \perp x_-^{-1}$).

    A morphism $G^+ \to H^+$ extends uniquely to $G \to H$ via $\phi(x) := \phi(x_+)\phi(x_-^{-1})^{-1}$.

    Proof: By uniqueness, $\phi(x_\pm) = \phi(x)_\pm$, so $\phi(x^{-1}) = \phi(x)^{-1}$; $x_-^{-1}(xy)_+y_-^{-1} = x_+y_+ \vee x_-^{-1}y_-^{-1}$ implies $\phi(xy)_+ = (\phi(x)\phi(y))_+$ and $\phi(xy)_- = (\phi(x)\phi(y))_-$, hence $\phi(xy) = \phi(x)\phi(y)$; by the first part, ϕ is a morphism.

12. The *polar* of a subset A is the convex lattice subgroup

$$A^\perp := \{\, x : |x| \wedge |a| = 1, \forall a \in A \,\}$$

    It is a dual map, i.e., $A \subseteq B^\perp \Leftrightarrow B \subseteq A^\perp$, hence $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$, $A \subseteq A^{\perp\perp}$, $A^\perp = A^{\perp\perp\perp}$. Also $A \cap A^\perp \subseteq \{\, 1 \,\}$, $(A \cup B)^\perp = A^\perp \cap B^\perp$.

    Proof: If $|x| \perp |a|$, $|y| \perp |a|$, then $|xy| \wedge |a| \leqslant |x||y||x| \wedge |a| = 1$; similarly for $|x \vee y|$; if $x \leqslant z \leqslant y$ then $|z| \wedge |a| \leqslant (|x| \vee |y|) \wedge |a| = 1$.

    If A is normal, then so is $A^\perp$ (since $\phi(A^\perp) = (\phi A)^\perp$ for any automorphism).

13. The Dedekind completion of an ordered group is a (lattice) group iff it is integrally closed, i.e., $\forall n \in \mathbb{N}, x^n \geqslant c \Rightarrow x \geqslant 1$.

    Proof: For $A \neq \varnothing, X$, let $x \in U(AL(A^{-1}))$, i.e., $Ay \leqslant 1 \Rightarrow Ay \leqslant x$, so $Ayx^{-1} \leqslant 1$ and by induction, $Ay \leqslant x^n$; hence $x \geqslant 1$, so $1^{LU} \subseteq (AL(A^{-1}))^{LU}$; but $Ay \leqslant 1 \Rightarrow Ay \subseteq L(1) = 1^{LU}$, so $A \cdot L(A^{-1}) = 1^{LU}$ (note $L(A^{-1}) = LUL(A^{-1})$). Conversely, if G is complete, let $a := \bigwedge_n x^n = 1 \wedge ax \leqslant ax$, so $x \geqslant 1$.

14. If G is complete, then $G = A^\perp \oplus A^{\perp\perp}$.

    Proof: Let $B := A^{\perp\perp}$; for any x, let $b := \bigvee(B^+ \wedge x_+) \in B^+$ and $c := x_+ b^{-1} \geqslant 1$; for all $a \in B^+$, $1 \leqslant a \wedge c = (ab \wedge x_+)b^{-1} \leqslant 1$ since $ab \in B^+$, so $c \in A^\perp$; similarly $x_- = b'c'$, so $x = bcb'c' = (bb')(cc') \in B \oplus A^\perp$.

15. There is an associated homogeneous topology generated by the open sets $B_y(a) := \{\, x : |x^{-1}a| < y \,\}$ where $y > 1$. In this topology,

    $$\mathcal{F} \to x \Leftrightarrow \quad \forall y > 1, \exists A \in \mathcal{F},\ z \in A \Rightarrow |z^{-1}x| < y$$

    $$x_n \to x \Leftrightarrow \quad \forall y > 1, \exists N,\ n \geqslant N \Rightarrow |x^{-1}x_n| < y$$

    The topology is T_0 when there is a sequence $y_n \searrow 1$.

Convex Lattice Subgroups

1. For any convex lattice subgroup, $x \in H \Leftrightarrow x_\pm \in H \Leftrightarrow |x| \in H$.

2. A subgroup is a convex lattice iff $x \wedge y, z \in H \Rightarrow x \wedge yz \in H$.

   Proof: $x \wedge y \leqslant x \wedge yz_+ \leqslant (x \wedge y)z_+ \in H$; so $(x \wedge yz_+)z_- \leqslant x \wedge yz_+z_- \leqslant x \wedge yz_+ \in H$.

3. If H, K are convex lattice subgroups then

   $$H \cap K = 1 \Leftrightarrow K \subseteq H^\perp \Leftrightarrow (1 \leqslant hk \Rightarrow 1 \leqslant h, k)$$

   In this case, $HK \cong H \times K$. (If $G = HK$, $H \cap K = 1$, then $G \cong H \times H^\perp$.)

   Proof: For $h \in H$, $k \in K$, $1 \leqslant |h| \wedge |k| \leqslant |h| \in H$, so $|h| \wedge |k| \in H \cap K = 1$, so h, k commute and $K \subseteq H^\perp$. In $H \times K \to HK$, $(h, k) \mapsto hk$; if $1 \leqslant hk$ then $1 \leqslant 1 \vee h^{-1} \leqslant 1 \vee k \in K$, so $1 \vee h^{-1} \in H \cap K = 1$ and $1 \leqslant h$. Conversely, if $h \in H \cap K$, then $hh^{-1} = 1$, so $h, h^{-1} \geqslant 1$.

4. The convex lattice subgroups of G form a (complete) Heyting algebra $\mathcal{C}(G)$ with $H \to K = \{\, x : \forall h \in H,\ |x| \wedge |h| \in K \,\}$ and a pseudo-complement $H^\perp = H \to 1$. A convex lattice subgroup is 'closed', i.e., $H^{\perp\perp} = H$, iff $H = K^\perp$.

5. The smallest convex lattice subgroup generated by A is

   $$[\![A]\!] = \{\, x : |x| \leqslant |a_1| \cdots |a_n|, \exists a_i \in A, n \in \mathbb{N} \,\} = \bigvee_{a \in A} [\![a]\!]$$

For any automorphism, $\phi[\![A]\!] = [\![\phi A]\!]$; if A is normal, so is $[\![A]\!]$.

$$\begin{aligned}[\![A]\!] \cap [\![B]\!] &= [\![|a| \wedge |b| : a \in A, b \in B]\!],\\ [\![A]\!] \vee [\![B]\!] &= [\![|a| \vee |b| : a \in A, b \in B]\!],\\ [\![A]\!]^{\perp} &= A^{\perp}\end{aligned}$$

In particular, $[\![a]\!] = \{ x : |x| \prec |a| \}$; $[\![a \vee b]\!] = [\![a]\!] \vee [\![b]\!] = [\![a, b]\!] = [\![|a||b|]\!]$, $[\![a \wedge b]\!] = [\![a]\!] \cap [\![b]\!]$. Every finitely generated convex lattice subgroup is principal, $[\![a_1, \ldots, a_n]\!] = [\![|a_1| \vee \cdots \vee |a_n|]\!]$. $[\![a]\!]$ are the compact elements in $\mathcal{C}(G)$.

Proof: Let B be the given set; for $x, y \in B$, $|xy| \leqslant |x||y||x|$, $|x^{-1}| = |x|$, $|x \vee y| \leqslant |x||y|$, and $x \leqslant z \leqslant y \Rightarrow |z| \leqslant |x| \vee |y|$, all being less than $\prod_i |a_i|$; $1 \leqslant |x| \leqslant \prod_{i=1}^n |a_i| \in [\![A]\!]$, so $|x|, x \in [\![A]\!]$ and $B \subseteq [\![A]\!]$. $a^{-1}[\![A]\!]a = \bigcap_{A \subseteq H} a^{-1}Ha = [\![a^{-1}Aa]\!] = [\![A]\!]$. If $|x| \leqslant \prod_i |a_i| \wedge |b_i| \leqslant \prod_i |a_i|, \prod_i |b_i|$; $|x| \leqslant \prod_i |a_i| \wedge \prod_j |b_j| \leqslant \prod_{ij} |a_i| \wedge |b_j|$. If $|x| \in [\![A \cup B]\!]$ then $|x| \leqslant \prod_i |a_i||b_i| \leqslant \prod_i (|a_i| \vee |b_i|)^2$. If $x \in A^{\perp}$ and $y \in [\![A]\!]$, then $|x| \wedge |y| \leqslant |x| \wedge |a_1| \cdots |a_n| = 1$.

6. For $\vee$-irreducible elements,

   (a) For any x, either $x_+ \perp a$ or $x_- \perp a$.

   (b) Independent $\vee$-irreducibles are orthogonal, i.e., $b \notin a^{\perp\perp} \Rightarrow a \perp b$ and $a^{\perp\perp} \cap b^{\perp\perp} = 1$.

   (c) $a^{\perp\perp}$ is linearly ordered (maximal in $\mathcal{C}(G)$).

   (d) $a^{\perp}$ is a minimal polar (and a minimal prime).

   Proof: For any x, either $a \wedge x_-^{-1} \leqslant a \wedge x_+ \leqslant a$, so $a \wedge x_-^{-1} = a \wedge x_-^{-1} \wedge x_+ = 1$, or $a \wedge x_+ = 1$. In particular, for $x, y \in a^{\perp\perp}$, either $(y^{-1}x)_+ \in a^{\perp} \cap a^{\perp\perp} = 1$ or $y^{-1}x \geqslant 1$. If $b \notin a^{\perp\perp}$ and $y \in a^{\perp}$, $b \wedge y \neq 1$, then $y \wedge a \wedge b = 1$ yet $a \wedge b$, $b \wedge y \in b^{\perp\perp}$, hence $a \wedge b = 1$. If $c \in a^{\perp\perp} \cap b^{\perp\perp}$ then $|c| \leqslant a, b$ so $|c| \leqslant a \wedge b = 1$. For any $y \in Y^{\perp} \subseteq a^{\perp\perp}$, $a^{\perp\perp} = y^{\perp\perp} \subseteq Y^{\perp} \subseteq a^{\perp\perp}$.

7. A convex lattice subgroup is said to be *prime* when it is $\wedge$-irreducible in $\mathcal{C}(G)$,

   $$P = H \cap K \Rightarrow P = H \text{ OR } P = K$$

   equivalently, P^{c} is closed under $\wedge$,

   $$x \wedge y \in P \Rightarrow x \in P \text{ OR } y \in P$$

   (or $x \wedge y = 1 \Rightarrow x \in P$ OR $y \in P$)

   (a) The cosets of P are linearly ordered.

   (b) The convex lattice subgroups containing P are linearly ordered.

Proof: If $x \wedge y \in P$, then $[\![P,x]\!] \cap [\![P,y]\!] = P \vee [\![x \wedge y]\!] = P$, so $P = [\![P,x]\!]$, say, and $x \in P$. Conversely, if $P = H \cap K$ and $h \in H \smallsetminus P$, $k \in K$, then $1 \leqslant |h| \wedge |k| \in H \cap K = P$, so $|k|, k \in P$, and $K \subseteq P$. $(x \wedge y)^{-1}(x \wedge y) = 1$, so $(x \wedge y)^{-1}x \in P$, say, i.e., $xP = (x \wedge y)P \leqslant yP$. If $P \subseteq H \cap K$, $h \in H$, $k \in K$ and $hP \leqslant kP$, say, then $h \leqslant kp$, so $1 \leqslant |h| \leqslant |kp| \in K$, hence $h \in K$; for any $x \in H$, $x \leqslant h^{-1}kp$, so $xP \leqslant h^{-1}kP$, hence $H \subseteq K$. If $x \wedge y = 1$ and $P \subseteq [\![P,x]\!] \subseteq [\![P,y]\!]$, then $|x| \leqslant |p_1||y| \cdots |p_n||y|$; by considering $|x| \wedge |x| \leqslant |p_1| \cdots |p_n|(|y| \wedge |x|)$, etc., it follows $|x| \leqslant |p|$, i.e., $x \in P$.

8. (a) Every subgroup containing P is a lattice.
   (b) The intersection of a chain of prime subgroups is prime.
   (c) The pre-image of a prime subgroup is prime.
   (d) Given a $\wedge$-sub-semi-lattice A, a maximal convex lattice subgroup in A^{c} is prime. Similarly, a convex lattice subgroup that maximally avoids being principal, is prime.

   Proof: Let $a \in H$, then since $a_+ \wedge a_-^{-1} = 1$, $a_+ \in P$ or $a_-^{-1} \in P$, so $a_+ = aa_-^{-1} \in H$; if $a, b \in H$ then $a \vee b = a(a^{-1}b)_+ \in H$. If $x \wedge y = 1$ then $\phi(x) \wedge \phi(y) = 1$, so $x \in \phi^{-1}P$, say. Given semi-lattice A, and $P = H \cap K$ but $P \neq H, K$, then $\exists a \in H \cap A$, $b \in K \cap A$; so $a \wedge b \in (H \cap K) \cap A = P \cap A = \varnothing$ a contradiction. If $H = [\![a]\!]$, $K = [\![b]\!]$ then $P = H \cap K = [\![a]\!] \cap [\![b]\!] = [\![a \wedge b]\!]$ contradicts that P is not principal.

9. A *regular* prime subgroup is one which is completely $\wedge$-irreducible,

$$P = \bigcap_i H_i \ \Rightarrow \ P = H_i, \ \exists i$$

$$\Leftrightarrow \ P \text{ is a maximal convex lattice subgroup in some } \{\, a \,\}^{\mathsf{c}}, (a \neq 1)$$

   Proof: For each $x \notin P$, there is a prime $Q_x \supseteq P$ which is maximal in x^{c}; so $P = \bigcap_{x \notin P} Q_x$ and $P = Q_a$ for some $a \notin P$. If $P = \bigcap_i H_i$, then $P \subset H_i \ \Rightarrow \ a \in H_i$, so $a \in \bigcap_i H_i = P$ unless $P = H_i$.

   (a) Every convex lattice subgroup is the intersection of regular primes: $H = \bigcap \{\, P_a : \text{regularprime}, 1 \leqslant a \notin H \,\}$.
   (b) Only 1 belongs to all primes.
   (c) $x \leqslant y \Leftrightarrow xP \leqslant yP$ for all regular P.

   Proof: $H \subseteq P_a$ since P_a is maximal in $\{\, a \,\}^{\mathsf{c}}$. If $x \notin H$ then $x_+ \notin H \subseteq P_{x_+}$, say (or $x_-^{-1} = (x^{-1})_+$), so $x \notin P_{x_+}$. If $xP \leqslant yP$ for all P, then $(x \vee y)P = yP$, so $(y^{-1}x)_+ = y^{-1}(x \vee y) \in P$; hence $(y^{-1}x)_+ = 1$, i.e., $x \leqslant y$.

10. For *minimal* primes, (every prime subgroup contains a minimal prime by Hausdorff's principle)

(a) P^{c} is a maximal $\wedge$-semi-lattice in 1^{c}.

(b) $\forall x \in P, \exists a \notin P, a \perp x$, i.e., $P = \bigcup_{a \notin P} a^{\perp}$.

Proof: If $1 \in A \subseteq P$ and A^{c} is a $\wedge$-semi-lattice, then A contains a maximal prime Q; then $P = Q = A$ by minimality.

If $x \in P$, so $|x| \in P$, then $P^{\mathsf{c}} \cup (|x| \wedge P^{\mathsf{c}})$ is a semi-lattice containing P^{c} properly, so $1 = |x| \wedge a$ for some $a \notin P$; conversely, if $x \in a^{\perp}$, $|a| \notin P$, then $|x| \wedge |a| = 1$ implies $|x|, x \in P$.

Structure of G

1. For a normal convex lattice subgroup H (*ideal*), G/H is again a lattice group with $xH \vee yH = (x \vee y)H$, $xH \wedge yH = (x \wedge y)H$. The ideals form a complete lattice $\mathcal{I}(G)$, as do the characteristic ideals (i.e., invariant under all automorphisms).

   For any sub-lattice-group L, LH is then a lattice group (since $xh \vee yk \in xH \vee yH = (x \vee y)H \subseteq LH$).

2. The isomorphism theorems hold: For any lattice subgroup L and ideals $H \subseteq K$,
$$G/\ker\phi \cong \phi G, \quad \frac{LH}{H} \cong \frac{H}{H \cap L}, \quad \frac{G/H}{K/H} \cong \frac{G}{K}$$
   Proof: The map $xH \mapsto \phi(x)$ preserves positivity: $(xH)_+ = x_+H \mapsto \phi(x_+) = \phi(x)_+$. Similarly, $L \to LH/H$, $x \mapsto xH$, and $xH \mapsto xK$ preserve positivity, hence are morphisms.

3. $G := \bigvee_i H_i \cong \sum_i H_i \;\Leftrightarrow\; H_i \trianglelefteq G$ AND $H_i \cap \bigvee_{j \neq i} H_j = 1 \;\Leftrightarrow\; H_i \cap H_j = 1 (i \neq j)$ (via the map $(x_i) \mapsto \prod_i x_i$).

   Proof: If $\prod_{i=1}^n x_i \geqslant 1$ then $x_j^{-1} \leqslant x_1 \cdots x_{j-1} x_j \cdots x_n =: y_j$; so $(x_j)_+^{-1} \leqslant (y_j)_+$, and $(x_j)_-^{-1} \in H_j \cap \bigvee_{i \neq j} H_i = 1$, i.e., $x_j \geqslant 1$. If $H_i \cap H_j = 1$, then $H_i \cap \bigvee_{j \neq i} H_j = \bigvee_{j \neq i}(H_i \cap H_j) = 1$.

4. For ideals H_i, $\frac{G}{\bigcap_i H_i} \lesssim \prod_i \frac{G}{H_i}$ via the morphism $x \mapsto (xH_i)$.

5. For a prime ideal, G/P is a linearly ordered space. A minimal proper ideal (atom of $\mathcal{I}(G)$) is linear.

   Proof: For any $x \in H \smallsetminus 1$ minimal, $H \cap x^{\perp} = 1$; so for $x, y \in H$, $x \wedge y = 1 \;\Rightarrow\; x = 1$ OR $y = 1$, hence H is linear.

6. The intersection of all prime ideals is an ideal, here called the 'radical' $\mathrm{rad}(G)$, since $a^{-1} \bigcap_i P_i a = \bigcap_i a^{-1} P_i a = \bigcap_i P_i$.

7. The splitting of a lattice group by ideals can continue until, perhaps, all such subgroups are simple.

   G is simple $\Leftrightarrow$ all of $G^+ \smallsetminus 1$ are conjugates of each other.

8. $[\![a]\!] = \{ x : |x| \prec |a| \}$ consists of $[\![x]\!]$ for each representative Archimedean class $|x| \prec |a|$. Extend the Archimedean classes by $[a] := \{ x : |x| \sim |a| \}$; then $[\![a]\!] = \bigcup_{|x|\prec|a|}[x]$.

9. A lattice group has no proper convex lattice subgroups iff it is an Archimedean linear group.

   Proof: For any $x \neq 1$, $[\![x]\!] = G$, so for all y, $|y| \prec |x|$; similarly $|x| \prec |y|$, so Archimedean. $\{ 1 \}$ is prime, so $G \cong G/1$ is linear.

10. Any atoms of $\mathcal{C}(G)$ are Archimedean linear and mutually orthogonal ($1 = [\![a]\!] \cap [\![b]\!] = [\![a \wedge b]\!]$). The sum of such atoms $\bigvee_i [\![a_i]\!] = \sum_i [\![a_i]\!]$ is here called the '$\mathcal{C}$-socle' of G (an ideal). Similarly, the sum of the atomic ideals is the $\mathcal{I}$-socle.

11. Another socle is the sum $\bigvee_a a^{\perp\perp}$ for a orthogonal $\vee$-irreducibles. A *group basis* of G is a maximal orthogonal set of proper $\vee$-irreducibles (so $E^\perp = 1$); there is a basis when the socle equals G.

    Proof: If $x > 1$ then $\exists y \in E, x \wedge y > 1$, else E is not maximal; $x \wedge y$ is $\vee$-irreducible. Conversely, let E be a maximal set of orthogonal $\vee$-irreducible elements. Then $x \in E^\perp$ and $x \geqslant e \geqslant 1$ imply $1 = e \wedge x \geqslant e = 1$.

12. A simple lattice group must either have trivial radical or have no proper prime ideals; it is either the sum of Archimedean linear groups or does not contain any. But otherwise, the simple lattice groups are not classified.

2.4 Representable Groups

are ordered groups that are embedded in a product of linearly ordered groups; equivalently, the radical is 1. For example, $\mathbb{Z}^n$, $G/\mathrm{rad}(G)$.

Proof: If $G \subsetneqq \prod_i X_i$ and π_i are the projections to X_i, then since 1 is prime, $\ker \pi_i$ are prime ideals; so $\mathrm{rad}(G) \subseteq \bigcap_i \ker \pi_i = \{ 1 \}$. Conversely, $G/1 \subsetneqq \prod_i G/P_i$.

1. (a) $(x \wedge y)^n = x^n \wedge y^n$

   (b) $x \wedge (y^{-1}xy) = 1 \;\Rightarrow\; x = 1$

   (c) $x \perp y \;\Rightarrow\; x \perp z^{-1}yz$.

   Proof: $(a_i)^n \wedge (b_i)^n = (a_i^n \wedge b_i^n) = (a_i \wedge b_i)^n$. If $x \wedge (y^{-1}xy) = 1$ then $a_i \wedge (b_i^{-1}a_ib_i) = 1$, so $a_i = 1$. $abab \wedge aa = (ab \wedge a)^2 \leqslant aba$, so $b \wedge a^{-1}b^{-1}a \leqslant 1$, in particular $b_+ \wedge a^{-1}b_-^{-1}a = 1$; for $b = xy^{-1}$, $x \wedge y = 1$, one gets $1 = x \wedge a^{-1}ya$.

2. Every prime contains a prime ideal.

   Proof: Let $N := \bigcap_x x^{-1}Px$ be the largest normal subgroup in P; if $a \wedge b = 1$ but $b \notin N$ then there is a y, $y^{-1}by \notin P$; so $x^{-1}ax \wedge y^{-1}by = 1$, and $x^{-1}ax \in P$ for all x, i.e., $a \in N$.

3. Polar and minimal prime subgroups are normal (i.e., ideals).

   Proof: A minimal prime subgroup satisfies $P = \bigcup\{ x^{\perp} : 1 \leqslant x \notin P \} = \bigcup\{ a^{-1}x^{\perp}a : 1 \leqslant x \notin P \} = a^{-1}Pa$. Conversely, if minimal primes are normal, then the radical is 1 (because every prime contains a minimal).

4. For any prime, either $xP \leqslant Px$ or $Px \leqslant xP$.

The *weakly abelian* lattice groups satisfy $\forall x \geqslant 1, y^{-1}xy \leqslant x^2$; then convex lattice subgroups are normal (if $x \in H$, $|a^{-1}xa| = a^{-1}|x|a \in H$).

2.4.1 Linearly Ordered Groups

when $G = G^+ \cup G^-$, i.e., every element is comparable to 1; equivalently, a lattice group without proper orthogonal elements $x \perp y \;\Rightarrow\; x = 1$ OR $y = 1$; or a lattice group all of whose convex subgroups are lattices. Every simple representable group is linearly ordered.

Examples:

- $\mathbb{Q}^+$ with multiplication
- Free group on an alphabet, e.g. $\_ < a^{-1}ba < b < aba^{-1} < a^{-1}bba < bb$ and pure braid groups.
- The lex product (lexicographic) of linear groups $\overleftarrow{\prod_i} G_i$, e.g. $\mathbb{Z}^n$ (not Archimedean).
- Torsion-less abelian groups can be made linear by embedding in $\mathbb{Q}^A$ (or consider a maximal set such that $P \cap P^{-1} = \{1\}$; if $1 \neq a \notin P \cup P^{-1}$ then the larger monoids generated by P and a or a^{-1} do not satisfy this condition; so $(xa^n)^{-1} = ya^m$, i.e., $a^{-(m+n)} = xy \in P$, as well as $a^{r+s} \in P$; hence $a^{(m+n)(r+s)} \in P \cap P^{-1}$, so $m = n = r = s = 0$ and $x = 1 = y$; thus $P \cup P^{-1} = X$.)
- $\mathbb{Z}^2$ with usual addition and $(0,0) \leqslant (x,y) \Leftrightarrow \alpha x \leqslant y$ ($\alpha \notin \mathbb{Q}$); e.g. $\alpha = \sqrt{2}$ gives $(0,0) < (-1,-1) < (0,1) < (-1,0) < (0,2) < (-1,1)$.
- Heisenberg group: $\mathbb{Z}^3$ with $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} * \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} := \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 + a_1 b_2 \end{pmatrix}$ and lexicographic ordering; a non-abelian linearly ordered group.
- Pure braid group (using its free group ordering).

1. Linear groups are either discrete or order-dense (since if $a < b$ is a gap so are $b^{-1}a < 1 < a^{-1}b$).

2. Every convex subgroup, including $\{1\}$, is prime ($x \wedge y = 1 \Rightarrow x = 1$ OR $y = 1$), so $\mathcal{C}(G)$ is a linear order. A linear group with a maximal convex subgroup is of the type $[\![a]\!]$.

3. If $[x^n, y^m] = 1$ ($m, n \neq 0$) then $[x, y] = 1$.

4. The center is an ideal.

5. The Archimedean relation $\prec$ is a coarser linear order on G: for any x, y either $x \prec y$ or $y \prec x$.

   The regular subgroup not containing a is $P_a = [1] \cup \cdots \cup [b] = \{ x : |x| \ll |a| \}$.

6. (Neumann) Every linearly ordered group is the image of a free linearly ordered group.

7. (Mal'cev) $\mathbb{Z}G$ is embedded in a division ring.

2.5 Completely Reducible Lattice Groups

are lattice groups whose socle equals the group; i.e., G is the sum of simple lattice groups. Every element has an irredundant decomposition $x = a_1 \vee \cdots a_n$ where $a_i \in X_i$.

The convex lattice subgroups satisfy ACC iff all such subgroups are principal iff G has a finite basis with each $a^{\perp\perp}$ satisfying ACC.

ACC lattice groups: they are complete, every element is compact.

2.6 Abelian Lattice Groups

They are representable since all prime subgroups are normal and $\text{rad}(G) = \bigcap_{P \text{ prime}} P = \{1\}$; thus every abelian lattice group is a product of linearly ordered abelian groups.

Hahn's theorem: Embedded in a lex product of $\mathbb{R}^A$ (where A is the number of Archimedean classes).

2.6.1 Archimedean Linear Groups

These are the simple abelian lattice groups.

Proposition 2

Hölder's embedding theorem

Every Archimedean linearly ordered group is embedded in $\mathbb{R}, +$.

PROOF: Fix $a > 1$ and let $L_x := \{ m/n \in \mathbb{Q} : a^m \leqslant x^n \}$, $U_x := \{ m/n \in \mathbb{Q} : a^m > x^n \}$, a Dedekind cut of $\mathbb{Q}$, i.e., $L_x \cup U_x = \mathbb{Q}$, $L_x \cap U_x = \varnothing$, $L_x < U_x$. Define $\phi : G \to \mathbb{R}, x \mapsto \sup L_x = \inf U_x$; given $m/n \in L_x$, $r/s \in L_y$, i.e.,

$a^m \leqslant x^n$, $a^r \leqslant y^s$, either $xy \leqslant yx$ when $a^{ms+nr} \leqslant x^{ns}y^{ns} \leqslant (xy)^{ns}$ or $yx \leqslant xy$ when $a^{nr+ms} \leqslant y^{ns}x^{ns} \leqslant (xy)^{ns}$; so $L_x + L_y \subseteq L_{xy}$; similarly, $U_x + U_y \subseteq U_{xy}$, so $\phi(xy) = \phi(x) + \phi(y)$. If $\phi(x) = 0$ then for all $m, n \geqslant 0$, $a^{-m} \leqslant x^n$, i.e., $1 \leqslant x \leqslant 1$. Hence ϕ is a 1-1 morphism.

□

Proposition 3

The only order-complete linearly ordered groups are 0, $\mathbb{Z}$ and $\mathbb{R}$.

Proof: Complete linear orders are Archimedean since $1 < x \ll y$ implies $\alpha := \sup_n x^n$ exists, so $\alpha x = x$, a contradiction. If $\mathbb{Z} \subset R \subset \mathbb{R}$, then there is $0 < \epsilon < 1$, hence R is order-dense in $\mathbb{R}$; its completion is $\mathbb{R}$.

□

1. They are therefore abelian and can be completed.

2. Any morphism between Archimedean linear groups is of the type $x \mapsto rx$ (as subgroups of $\mathbb{R}$).

   Proof: For $\phi \neq 0$, let $\phi(a) > 0$; if $\frac{\phi(x)}{\phi(a)} < \frac{m}{n} < \frac{x}{a}$ then $ma < nx$ so $m\phi(a) < n\phi(x)$ a contradiction; so $\phi(x)/x = r := \phi(a)/a$.

Ordered Rings

3 Ordered Modules and Rings

An **ordered ring** is a unital ring with an order such that $+$ is monotone, and $*$ is monotone with respect to positive elements, i.e., $a, b \geqslant 0 \;\Rightarrow\; ab \geqslant 0$.

An **ordered module** is an ordered abelian group X acted upon by an ordered ring R such that for $a \in R$, $x \in X$,

$$a \geqslant 0, x \geqslant 0 \;\Rightarrow\; ax \geqslant 0$$

Hence $a \geqslant 0$ AND $x \leqslant y \;\Rightarrow\; ax \leqslant ay$; similarly, $a \leqslant b$ AND $x \geqslant 0 \;\Rightarrow\; ax \leqslant bx$; if $a \leqslant 0$ then $ax \geqslant ay$ (since $\pm a(y - x) \geqslant 0$). For rings, $a \geqslant 0$ AND $b \leqslant c \;\Rightarrow\; ba \leqslant ca$.

The morphisms are the maps that preserve $+$, $\cdot$, $\leqslant$; module morphisms need to preserve the action $T(ax) = aTx$. An *ordered algebra* is an ordered ring which is a module over itself (acting left and right).

X^+ is closed under $+, \cdot$, and uniquely determines the order on X, $x \leqslant y \Leftrightarrow y - x \in X^+$; any subset $P \subseteq R$ such that $P + P \subseteq P$, $PP \subseteq P$ and $P \cap (-P) = 0$ defines an order on R. (For X, replace with $R^+P \subseteq P$.)

Examples:

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ with their linear orders. $\mathbb{Z}$ has a unique linear order ($1 \nleqslant 0$, see later). $\mathbb{Q}$ has a unique linear order that extends that of $\mathbb{Z}$: for $n > 0$, $\frac{1}{n} + \cdots + \frac{1}{n} = 1$, so $\frac{1}{n} > 0$; so $m/n > 0$ for $m, n > 0$.
- $\mathbb{Z}$ with $2\mathbb{N} \geqslant 0$; $\mathbb{Q}$ with $\mathbb{N} \geqslant 0$; $\mathbb{C}$ with $\mathbb{R}^+ \geqslant 0$.
- $\mathbb{Z}_2 \times \mathbb{Z}$ with $(0, 1), (1, 2) \geqslant 0$.
- $\mathbb{Q}(\sqrt{2})$ with $0 < 1$ but $\sqrt{2}$ not comparable to 0 or 1.
- A commutative *formally real* ring ($\sum_n^N x_n^2 = 0 \;\Rightarrow\; x_n = 0$) has a natural (minimal) positive cone $P := \sum \prod R^2$ (finite terms). Equivalently, squares are positive and there are no nilpotents. If R is formally real, then so are $R[x, y, \ldots]$, R^A, subrings (e.g. $C(R)$).

  More generally, any ring with the property that finite sums of terms $a_1 \cdots a_{2n}$, where each a_i occurs an even number of times, can be zero only if each product is zero, has an order whose positives consist of such sums (such as squares).
- *Scaled ring*: For any ordered ring/module, pick any invertible central positive element λ, and let $a * x := \lambda ax$; the new identity is λ^{-1}.
- Any module with the trivial order $X^+ = 0$. Every finite module, being a finite group, can only have this order.

- $\text{Hom}(X)$, the morphisms of a commutative ordered monoid, with $0 \leqslant \phi \Leftrightarrow 0 \leqslant \phi(x), \forall x \geqslant 0$, AND $\phi(x) \leqslant 0, \forall x \leqslant 0$. It is pre-ordered, but ordered when $X = X^+ + X^-$. Every ordered ring is embedded in such a ring, via the map $a \mapsto \phi_a$ where $\phi_a(x) := ax$.

Sub-modules (e.g. left ideals) and sub-rings are automatically ordered; in particular the generated sub-modules and sub-rings $[\![A]\!]$.

Products of ordered modules (rings) $X \times Y$ with

$$(x, y) \geqslant 0 \Leftrightarrow x \geqslant 0 \text{ AND } y \geqslant 0$$

and functions X^A , with

$$f \geqslant 0 \Leftrightarrow f(x) \geqslant 0 \; \forall x \in A$$

are again ordered modules (rings). But $R \overset{\leftarrow}{\times} S$ is not, e.g. $(0, 1), (1, -1) > 0$ yet $(0, 1)((1, -1) = (0, -1) < 0$.

Matrices $M_n(R)$ with $0 \leqslant T \Leftrightarrow T_{ij} \geqslant 0, \forall i, j$ (i.e., $M_n(R^+)$.

Polynomials $R[x]$ with $R[x]^+$ consisting of polynomials with (i) $p(a) \geqslant 0$ for all $a \in R$, (ii) all coefficients are positive, $R^+[x]$, or (iii) lex ordering: lowest order term is positive; apart from (iv) $p = \sum_i q_i^2$ when formally real; note (iv) $\Rightarrow$ (i) $\Rightarrow$ (iii). In $\mathbb{Z}[x]$, x satisfies (ii) but not (i) or (iv), $x^2 - x + 1$ satisfies (i) but not (ii) or (iv).

Series $R[[x]]$ and Laurent series $R((x))$ with lex ordering.

Group Algebras: More generally, $R[\mathcal{C}]$ with convolution and $R[\mathcal{C}]^+ = R^+[\mathcal{C}]$.

If R acts on X and $\phi : S \to R$ is a morphism, then S acts on X by $s \cdot x := \phi(s)x$.

1. | | X^+ | X^- |
   |---|---|---|
   | R^+ | $+$ | $-$ |
   | R^- | $-$ | $+$ |

   So $a \in R^{\pm} \Rightarrow a^2 \geqslant 0$ and $0 \leqslant a \leqslant b \Rightarrow a^2 \leqslant b^2$. In particular $1 \not< 0$ (else $1 < 0 \Rightarrow 1^2 > 0$); for any idempotent $e \not< 0$, $e \not> 1$. But squares need not be positive, e.g. in $\mathbb{Z}[x]$, $(x-1)^2 = x^2 - 2x + 1$ is unrelated to 0; in $M_2(\mathbb{Z})$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -I < 0$.

2. $0 \leqslant a \leqslant b$ AND $0 \leqslant x \leqslant y \Rightarrow ax \leqslant by$.

   In particular, $0 \leqslant a, b \leqslant 1 \Rightarrow ab \leqslant 1$.

   $a \geqslant 1$ AND $x \leqslant y \Rightarrow ax \leqslant ay$ (since $(a-1)(y-x) \geqslant 0$); $a, b \geqslant 1 \Rightarrow ab \geqslant 1$.

   If $ab = 0$ for $a, b \geqslant 0$, then $(a \wedge b)^2 = 0$.

   If $x + y = 0$ with $x, y \geqslant 0$ then $x = 0 = y$, i.e., $x > 0, y \geqslant 0 \Rightarrow x + y > 0$.

   Note that $ax \geqslant 0, x > 0 \not\Rightarrow a \geqslant 0$.

3. Convex sub-modules give ordered-module quotients with

$$0 + Y \leqslant x + Y \Leftrightarrow \exists y \in Y,\ x + y \geqslant 0$$

   Similarly, convex ideals for rings. For a discrete module, all sub-modules are convex.

   A sub-module is convex iff $x, y \geqslant 0, x + y \in Y \Rightarrow x, y \in Y$. For example, $\text{Annih}(x)$; more generally $[M : B] := \{ a \in R : aB \subseteq M \}$ when M is a convex sub-module and $B \geqslant 0$.

   A convex ideal of $M_n(R)$ is of the form $M_n(I)$ with I a convex ideal.

4. Morphisms pull convex sub-modules (ideals) to convex sub-modules (ideals) $T^{-1}M$, in particular $\ker T = T^{-1}0$.

5. When 1 and 0 are incomparable, one can distinguish the *quasi-positive* elements of X

$$a \geqslant 0 \Rightarrow ax \geqslant 0$$

   They form an upper-closed sub-semi-module that contains X^+; and closed under $\cdot$ for R.

   For any quasi-positive idempotent, eRe is a subring with $(eRe)^+ = eR^+e$.

Types of Ordered Modules/Rings:

- An ordered ring is *reduced* when it has no non-trivial positive/negative nilpotents, i.e., $a > 0 \Rightarrow a^2 > 0$.
- It is an ordered *domain* when it has no non-trivial positive/negative zero divisors, i.e., $a, b > 0 \Rightarrow ab > 0$. Ordered domains are reduced.
- An ordered module is *simple* when it contains no proper convex sub-modules. A left-simple ordered ring is an ordered domain, since $ab = 0, b > 0 \Rightarrow \text{Annih}(b) = R$.
- It is *Archimedean* when $X, +$ is an Archimedean group. An Archimedean ring with $0 < 1$ is left-simple, since if $0 \neq a \in I$ then $1 \leqslant n|a| \in I$ and $1 \in I$. Simple ordered modules, acted on by rings with $R \prec 1$, are Archimedean, as $\{ x : x \prec y \}$ is a convex sub-module.

3.0.2 Lattice Ordered Rings/Modules

Hence $X, +$ is an abelian lattice group,

$$x + y \vee z = (x + y) \vee (x + z)$$

Morphisms must preserve the operations $+$, $\cdot$, $\vee$. Note that an isomorphism is a bijective morphism.

Examples:

- $\mathbb{Z}[\sqrt{2}]$ with $a+b\sqrt{2}\geqslant 0 \Leftrightarrow b\leqslant a\leqslant 2b$ (more generally, any angled sector less than π).
- $\mathbb{Z}^2$ with standard $+,\leqslant$ and (i) $(a,b)(c,d):=(ac+bd,ad+bc)$, (ii) $(a,b)(c,d):=(ac,ad+bc+bd)$.
- Any abelian lattice group acted upon by its ring of automorphisms, with $\phi\geqslant 0 \Leftrightarrow \phi G^+\subseteq G^+$.
  The bounded morphisms $\mathrm{Hom}_B(X)$ of a complete lattice group.
- $M_2(\mathbb{Z})$ with $\begin{pmatrix}a&b\\c&d\end{pmatrix}\geqslant 0 \Leftrightarrow 0\leqslant c\leqslant a, 0\leqslant d\leqslant b$. Then $0\not\leqslant 1$.
- The infinite matrices over $\mathbb{Z}$ with a finite number of non-zero entries; the subring of upper triangular matrices.
- Group algebras $\mathbb{F}[G]$, with $\mathbb{F}[G]^+:=\mathbb{F}^+[G]$.

Products $X\times Y$ and functions X^A are again lattice ordered. Matrices $M_n(R)$ are lattice ordered rings when R is a lattice ordered ring.

Every subset generates a sub-lattice-ring $[\![A]\!]$.

1. Recall from abelian lattice groups: $x_+:=x\vee 0$, $x_-:=x\wedge 0$,

$$\begin{array}{rcl} x=x_++x_- & (x\vee y)_\pm=x_\pm\vee y_\pm & (-x)_+=-x_- \\ |x|=x_+-x_-=x\vee(-x) & |x+y|\leqslant|x|+|y| & |-x|=|x| \\ -|x|\leqslant x\leqslant|x| & |nx|=n|x| & |x\vee y|\leqslant|x|+|y| \\ -(x\vee y)=(-x)\wedge(-y) & x\vee y+x\wedge y=x+y & x\wedge y=0=x\wedge z \Rightarrow x\wedge(y+z)=0 \\ & n(x\vee y)=\begin{cases}nx\vee ny, & n\geqslant 0\\ nx\wedge ny, & n\leqslant 0\end{cases} & nx\geqslant 0\Leftrightarrow x\geqslant 0 \\ x\vee y=(x-y)_++y & & nx=0\Leftrightarrow x=0 \end{array}$$

   If $|x|\wedge|y|=0$ then $(x+y)_\pm=x_\pm+y_\pm$ and $|x+y|=|x|+|y|=|x|\vee|y|$.
   Morphisms: $(Tx)_+=Tx_+$, $T|x|=|Tx|$.

2. If $a\geqslant 0$ then $a(x\vee y)\geqslant ax\vee ay$, $a(x\wedge y)\leqslant ax\wedge ay$;
   If $a\leqslant 0$ then $a(x\vee y)\leqslant ax\wedge ay$, $a(x\wedge y)\geqslant ax\vee ay$.
   If $x\geqslant 0$ then $(a\vee b)x\geqslant ax\vee bx$, $(a\wedge b)x\leqslant ax\wedge bx$;
   If $x\leqslant 0$ then $(a\vee b)x\leqslant ax\wedge bx$, $(a\wedge b)x\geqslant ax\vee bx$.
   If $a,a^{-1}>0$ then $a(x\vee y)=ax\vee ay$ and $a(x\wedge y)=ax\wedge ay$, since $ax,ay\leqslant z\Leftrightarrow x,y\leqslant a^{-1}z$. Note that $\begin{pmatrix}1&1\\0&1\end{pmatrix}>0$ but $\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}\not\geqslant 0$.

3. $|ax|\leqslant|a||x|$
   Proof:

$$\begin{aligned} ax=(a_++a_-)(x_++x_-) &\leqslant a_+x_+-a_+x_--a_-x_++a_-x_-=|a||x| \\ &\geqslant -a_+x_++a_+x_-+a_-x_+-a_-x_-=-|a||x| \end{aligned}$$

4. ℓ-sub-modules are the convex sub-lattice-modules; they are the kernels of morphisms, and X/Y is a lattice ordered module; similarly for ℓ-ideals and rings.

   A sub-lattice-module is convex iff $x \in Y, |y| \leqslant |x| \Rightarrow y \in Y$.

   An ℓ-ideal which is a prime subgroup gives a quotient which is linearly ordered.

5. First Isomorphism theorem: If T is a module morphism, then

$$X/\ker T \cong \operatorname{im} T \quad \text{via } x \mapsto Tx.$$

   Proof: If $0 \leqslant Tx$ then $Tx = (Tx)_+ = Tx_+$, so $Tx_- = 0$ and $x + \ker T \geqslant \ker T$. An order-isomorphism is a $\vee$-isomorphism.

6. If $X = M + N$, both ℓ-submodules, then

$$\frac{X}{M \cap N} \cong \frac{X}{M} \times \frac{X}{N}$$

   For ℓ-sub-modules, $\frac{X}{\bigcap_i Y_i} \subsetneq \prod_i \frac{X}{Y_i}$ via $x \mapsto (x + Y_i)$.

7. A coarser relation than the Archimedean one is $|x| \leqslant |a||y|$ for some $a \in R$. Let

$$|A \cdot Y| := \{\, x \in X : |x| \leqslant |a_1||y_1| + \cdots + |a_n||y_n|, a_i \in A, y_i \in Y, n \in \mathbb{N} \,\}$$

   Note that $|\sum_i a_i y_i| \leqslant \sum_i |a_i||y_i|$, so $A \cdot Y \subseteq |A \cdot Y|$.

   The ℓ-sub-module generated by a subset is $\widehat{[\![Y]\!]} = |R \cdot Y|$, in particular if Y is an sub-lattice-module then

$$\widehat{[\![Y]\!]} = \{\, x \in X : |x| \leqslant |a||y|, a \in R, y \in Y \,\}$$

   e.g. $\widehat{[\![y_1, y_2]\!]} = \widehat{[\![|y_1| + |y_2|]\!]}$ so finitely generated modules are one-generated; $M \vee N = \{\, x : |x| \leqslant |a|(|y| + |z|), a \in R, y \in M, z \in N \,\}$. Similarly, the generated convex ideal is

$$\widehat{\langle A \rangle} = \{\, b : |b| \leqslant |r|(|a_1| + \cdots + |a_n|)|s|, r, s \in R, a_i \in A, n \in \mathbb{N} \,\}$$

   The ℓ-sub-modules form a complete distributive lattice.

8. The ℓ-annihilator of a subset $B \subseteq X$ is

$$\operatorname{Annih}_\ell(B) := \{\, a \in R : |a||x| = 0, \forall x \in B \,\} \subseteq \operatorname{Annih}(B)$$

   is a left ℓ-ideal of R. Similarly the ℓ-zero-set of $A \subseteq R$ is

$$\operatorname{Zeros}_\ell(A) = \{\, x \in X : |a||x| = 0, \forall a \in A \,\} \subseteq \operatorname{Zeros}(A)$$

   is a convex lattice-subgroup (but not a module).

9. For the lattice of ℓ-ideals,

   (a) I is an *ℓ-nilpotent* ideal iff $|I^n| = 0$; it is nilpotent. If I is a nilpotent left ℓ-ideal, then so is $\widehat{\langle I \rangle} = |I \cdot R|$.

   (b) I is an *ℓ-nil* ideal iff for $x \in I$, $|x|$ is nilpotent.

   (c) S is an *ℓ-semi-prime* ideal iff $|I \cdot J| \subseteq S \;\Rightarrow\; I \cap J \subseteq S$ iff $|x|R|x| \subseteq S \;\Rightarrow\; x \in S$. A convex semi-prime ideal is ℓ-semi-prime.

   (d) P is an *ℓ-prime* ideal iff $|I \cdot J| \subseteq P \;\Rightarrow\; I \subseteq P$ OR $J \subseteq P$ iff $|x|R|y| \subseteq P \;\Rightarrow\; x \in P$ OR $y \in P$. A convex prime ideal is ℓ-prime.

   (e) P is an *ℓ-primitive* ideal iff P is the ℓ-core $\mathrm{Annih}_\ell(R/I)$ (the largest ℓ-left-ideal) of some maximal ℓ-left-ideal I.

10. Convex Radicals for Rings:

    $\mathrm{Nil}_\ell := \sum \ell$-nil ideals,

    $\mathrm{Nilp}_\ell := \{\, x : |x| \text{ supernilpotent} \,\} = \sum \ell$-nilpotent ideals.

    $\mathrm{Prime}_\ell := \bigcap\{\, P : \ell\text{-prime ideal} \,\}$, (the smallest ℓ-semi-prime)

    $\mathrm{Jac}_\ell := \{\, x : |x| \text{ quasi-nilpotent} \,\}$

    $$\mathrm{Nilp}_\ell \subseteq \mathrm{Prime}_\ell \subseteq \mathrm{Nil}_\ell \subseteq \mathrm{Jac}_\ell$$

    Proof: Same as for rings, e.g. $\mathrm{Prime}_\ell \subseteq \mathrm{Nil}_\ell$: if $|x|$ is not nilpotent then there is an ℓ-prime which is maximal in not containing any $|x|^n$; so if $I, J \not\subseteq P$ then $|x|^n \in |I+P|$, $|x|^m \in \||J+P|$, hence $|x|^{n+m} \in |I+P| \cdot |J+P| \subseteq |(I+P)\cdot(J+P)| = |I \cdot J + P|$, $\therefore I \cdot J \not\subseteq P$, so P is ℓ-prime and $|x| \notin P$.

11. *Semi-prime Ordered Rings*: when $\mathrm{Prime}_\ell(R) = 0$, equivalently, it contains no proper ℓ-nilpotent ideals, $|I^n| = 0 \;\Rightarrow\; I = 0$, or 0 is ℓ-semi-prime

    $$|a|R|a| = 0 \;\Rightarrow\; a = 0.$$

    $R/\mathrm{Prime}_\ell \subsetneqq \prod$ prime ordered rings.

12. *Prime Ordered Rings*: when 0 is ℓ-prime, i.e., $|I \cdot J| = 0 \;\Rightarrow\; I = 0$ OR $J = 0$; equivalently, for any left ℓ-ideal, $\mathrm{Annih}_\ell(I) = 0$. Examples include $M_n(R)$ when R is a linearly ordered division ring.

13. A reduced ordered ring is embedded in a product of domains $\prod_M R/M$ where M are the minimal ℓ-primes. A reduced prime ordered ring is a domain.

14. If R is commutative, then $ab = (a \vee b)(a \wedge b)$, so $a \wedge b = 0 \;\Rightarrow\; ab = 0$; in particular, $a^2 = (a_+ + a_-)^2 = a_+^2 + a_-^2 \geqslant 0$, including $1 \geqslant 0$. Thus a commutative lattice ordered ring without nilpotents is formally real.

15. Recall the topology generated by $B_y(x)$ for $y > 0$. A coarser topology is that generated by $B_{ay}(x)$ for fixed y and $a \in R^+$.

3.1 Lattice Modules/Rings

A **lattice module** is a lattice-ordered module acted upon by a lattice-ordered ring such that

$$a \geqslant 0 \;\Rightarrow\; a(x \vee y) = ax \vee ay$$
$$x \geqslant 0 \;\Rightarrow\; (a \vee b)x = ax \vee bx$$

The morphisms need to preserve $+$, $*$, $\vee$. A **lattice ring** is a lattice-ordered ring which is a lattice module over itself.

Thus $R^+, *$ is a lattice monoid.

Examples:

- $\mathbb{Z}^2$, $\mathbb{Q}^n$, e.g. $(1,0)(0,1) = (0,0)$.
- *Vector lattices*: a lattice ordered module acted upon by a linearly ordered division ring, since $a \vee b = a$ or b, and $a > 0 \;\Rightarrow\; a^{-1} > 0$. A *Riesz space* is a vector lattice over $\mathbb{R}$.
- Archimedean lattice ordered rings, since $x \wedge y = 0 \;\Rightarrow\; ax \wedge y \leqslant nx \wedge y \leqslant n(x \wedge y) = 0$.

Sub-lattice-rings, images are again lattice-rings. Products, R^A, its sub-lattice ring $C(X)$ when X is a T_2 space; but not matrices $M_n(R)$ or $R[G]$.

1. $a \geqslant 0 \;\Rightarrow\; a(x \wedge y) = ax \wedge ay$, $x \geqslant 0 \;\Rightarrow\; (a \wedge b)x = ax \wedge bx$
   $a \leqslant 0 \;\Rightarrow\; a(x \vee y) = ax \wedge ay$, $x \leqslant 0 \;\Rightarrow\; (a \vee b)x = ax \wedge bx$.
   $ax \wedge by \leqslant (a \vee b)(x \wedge y) \leqslant ax \vee by$
2. Equivalently,
   (a) $|ax| = |a||x|$,
   (b) $(ax)_+ = a_+x_+ + a_-x_-$, $(ax)_- = a_+x_- + a_-x_+$
   (c) $a \geqslant 0 \;\Rightarrow\; ax_+ \wedge (-ax_-) = 0$
   $x \geqslant 0 \;\Rightarrow\; a_+x \wedge (-a_-x) = 0$
   (d) $a \geqslant 0$ AND $x \wedge y = 0 \;\Rightarrow\; ax \wedge ay = 0$,
   $x \geqslant 0$ AND $a \wedge b = 0 \;\Rightarrow\; ax \wedge bx = 0$
   (e) $a, b \geqslant 0$ AND $x \wedge y = 0 \;\Rightarrow\; ax \wedge by = 0 = xa \wedge yb$ (for rings)
   $x, y \geqslant 0$ AND $a \wedge b = 0 \;\Rightarrow\; ax \wedge by = 0$

   Proof: (e) $0 \leqslant ax \wedge by \leqslant (a \vee b)(x \wedge y) = 0$. (e) $\Rightarrow$ (d) $\Rightarrow$ (c) trivial; $ax = (a_+ + a_-)(x_+ + x_-) = (a_+x_+ + a_-x_-) + (a_+x_- + a_-x_+)$; but $(a_+x_+ + a_-x_-) \perp (a_+x_- + a_-x_+)$, so $(ax)_+ = a_+x_+ + a_-x_-$, etc.; hence $|ax| = (ax)_+ - (ax)_- = |a||x|$. For $a \geqslant 0$, $2(ax)_+ = ax + a|x| = 2ax_+$, so $a(x \vee y) = a(x-y)_+ + ay = ax \vee ay$; similarly for $(a \vee b)x = ax \vee bx$.

   Every lattice-ordered ring contains a lattice ring, namely $\{\, a \in R : x \wedge y = 0 \;\Rightarrow\; |a|x \wedge y = 0 = x|a| \wedge y \,\}$.

3. Hence $\text{Annih}_\ell(B) = \text{Annih}(B)$, $\text{Zeros}_\ell(A) = \text{Zeros}(A)$. If M is a submodule, then $\text{Annih}(M)$ is an ℓ-ideal; if I is an ideal, then $\text{Zeros}(I)$ is an ℓ-submodule.

4. $|1|x = x = 1_+x$, $1_-x = 0$

   Proof: $(1 \wedge 0)x = x_+ \wedge 0 + x_- \vee 0 = 0$.

5. $A^\perp$ is an ℓ-submodule (or ℓ-ideal) and $\widehat{[\![A]\!]} \cap A^\perp = 0$; $\widehat{[\![A]\!]}^\perp = A^\perp$.

   Proof: If $x \in \widehat{[\![A]\!]} \cap A^\perp$, then $|x| \wedge |x| \leqslant r(|a_1| + \cdots + |a_n|) \wedge |x| = 0$.

6. If $v \wedge w = 0$ for $v \in V$, $w \in W$, then $\widehat{[\![V]\!]} \cap \widehat{[\![W]\!]} = 0$.

   For a vector lattice, if $v_i \wedge v_j = 0$ (non-zero) then $\sum_i a_i v_i \geqslant 0 \Leftrightarrow a_i \geqslant 0$. Thus v_i are linearly independent. Hence a finite dimensional vector lattice has a finite group basis.

   Proof: If $a_1 \leqslant 0$, then $0 \leqslant (-a_1v_1) \wedge v_1 \leqslant (a_2v_2 + \cdots + a_nv_n) \wedge v_1$, so $-a_1v_1 \wedge v_1 = 0$ and $a_1 = 0$.

7. A convex sub-module of $X \times Y$ is of the form $M \times N$ with M, N convex sub-modules.

8. $R/\text{Annih}(x) \cong Rx$ for $x \geqslant 0$, via $a \mapsto ax$.

9. An indecomposable lattice module is linearly ordered.

   Proof: $X = x_+^{\perp\perp} \oplus x_+^\perp$, hence either $x_+ \in x_+^{\perp\perp} = 0$ or $x_- \in x_+^\perp = 0$.

10. Lattice modules and rings can be embedded in a product of linearly ordered modules/rings. (Equivalent to definition.)

    Proof: The radical is 0 (as an abelian lattice group), so $X \subsetneqq \prod_P X/P$ via $x \mapsto (x+P)_{P \in \mathcal{P}}$; the embedding is a lattice ring morphism. An ℓ-prime lattice ring is linearly ordered: $\widehat{\langle x_+ \rangle} \cdot \widehat{\langle x_- \rangle} \subseteq \widehat{\langle x_+ \rangle} \cap \widehat{\langle x_- \rangle} = 0$, so $x_+ = 0$ or $x_- = 0$.

11. $M_n(R)$ acts trivially on a lattice module ($Ax = 0$), unless $n = 1$.

    Proof: Suppose $M_n(R)$ acts on a lattice module, hence on a linearly ordered module X; then $E_{1j}x \leqslant E_{2j}x$, say, so multiplying by E_{i1} and E_{i2} gives $E_{ij}x = 0$.

    Lattice Rings

12. $0 \leqslant 1$, so R contains $\mathbb{Z}$ (unless $R = 0$), since $1_+ = 1_+1 = 1$.

13. Let $a_\oplus := a \vee 1$, $a_\ominus := a \wedge 1$, for $a \geqslant 0$. Then $a = a_\oplus a_\ominus$.

14. $a \perp b \;\Rightarrow\; ab = 0$. In particular $a_+a_- = 0$ and $1^\perp = 0$.

    Proof: $a \wedge b = 0 \;\Rightarrow\; ab \wedge b = 0 \;\Rightarrow\; ab \wedge ab = 0$.

    The converse holds iff the lattice ring is reduced (since $0 = |ab| \geqslant (|a| \wedge |b|)^2 \;\Rightarrow\; a \perp b$).

15. Squares are positive: $a^2 = |a|^2 \geqslant 0$ since $a^2 = (a_+ + a_-)^2 = a_+^2 + a_-^2 \geqslant 0$.
    (a) If a is invertible, then $a > 0 \Rightarrow a^{-1} > 0$ (since $a^{-1} = (a^{-1})^2 a \geqslant 0$).
    (b) $ab + ba \leqslant a^2 + b^2$ since $(a-b)^2 \geqslant 0$.
    (c) Idempotents satisfy $0 \leqslant e \leqslant 1$. Any proper idempotent decomposes $X = eX \oplus (1-e)X$ (eX is convex since $0 \leqslant y \leqslant ex \Rightarrow (1-e)y = 0$).
16. (a) $|a^n| = |a|^n$ (possibly $n < 0$)
    (b) $|a|^n \leqslant 1 \Leftrightarrow |a| \leqslant 1$, i.e., $-1 \leqslant a^n \leqslant 1 \Rightarrow -1 \leqslant a \leqslant 1$
    $|a|^n \geqslant 1 \Leftrightarrow |a| \geqslant 1$
    (c) Nilpotents satisfy $|a| \ll 1$, since na is also nilpotent.
17. $A^\perp + B^\perp \subseteq (AB)^\perp$
18. Idempotents are central.

    Proof: Embed in linear ordered rings; then $e = (0 \text{ or } 1)$ (see later) so commutes.
19. As Archimedean classes, $ab - ba \ll a^2 + b^2$. So an Archimedean lattice ring is commutative.

    Proof: Assume a linear order, $0 \leqslant a \leqslant b$; then $nb = ka + r$ with $0 \leqslant r < a$; so $n(ab - ba) = a(nb) - (nb)a = [a, r]$, so $n|[a, b]| = |[a, r]| \leqslant 2a^2 \leqslant a^2 + b^2$.
20. If $A \geqslant 0$ then its centralizer $Z(A)$ is a sub-lattice-ring, e.g. the center $Z(R) = Z(R^+)$.
21. If I is a convex left ideal then its core $[I : R] = \{ a \in R : aR \subseteq I \} \subseteq I$ is an ℓ-ideal.
22. $\text{Nilp}_\ell = \text{Nil}_\ell$, $\text{Nil}_n := \{ a : a^n = 0 \}$ are ℓ-nilpotent ideals.

    Proof: Assume linearly ordered; $a^m = 0 = b^n$, $0 \leqslant a \leqslant b \Rightarrow (a+b)^n \leqslant (2b)^n = 2^n b^n = 0$; $|ra| \leqslant |ar| \Rightarrow 0 \leqslant |ra|^n \leqslant |ar|^n \leqslant |a||ra|^{n-1}|r| \leqslant \cdots \leqslant |a|^n |r|^n = 0$, similarly for $|ar| \leqslant |ra|$. If $|b| \leqslant |a|$ then $0 \leqslant |b^n| = |b|^n \leqslant |a|^n = |a^n| = 0$ hence convex. If $a \in \text{Nil}_\ell$, then $a \in \text{Nil}_n$ for some n, so $a \in \sum_n \text{Nil}_n \subseteq \text{Nilp}_\ell$.
23. (Johnson) $R/\text{Nil}_\ell \subsetneq \prod_n R_n$ linear domains.
24. Archimedean vector lattices over a field are isomorphic to $\mathbb{R}^n$.

3.2 Linearly Ordered Rings

Equivalently, a lattice-ordered ring with $x \wedge y = 0 \Rightarrow x = 0$ OR $y = 0$. They are lattice rings since $a(x \vee y) = ax = ax \vee ay$ (say).

Examples:

- $\mathbb{Z}^2$ or $\mathbb{Q}^2$ with lex ordering and $(a, b)(c, d) := (ac, ad + bc)$ or $(a, b)(c, d) := (ad + bc, bd)$; non-Archimedean.

- Any commutative lattice-ordered domain, since $x \wedge y = 0 \;\Rightarrow\; xy = 0 \;\Rightarrow\; x = 0$ or $y = 0$.
- $R[x]$, $R[[x]]$, $R((x))$ with lex ordering. The subring of terms $\sum_{n=-N}^{M} a_n x^n$.
- Ring of fractions is also linearly ordered (when commutative)

$$a/b \leqslant c/d \Leftrightarrow ad \leqslant bc \quad (\text{for } b, d \geqslant 0)$$

Hence a commutative linearly ordered ring extends to a linearly ordered field, e.g. $\mathbb{Z}$ to $\mathbb{Q}$.

1. Equivalently, they are the indecomposable lattice rings (no proper idempotents).

   Proof: For any idempotent, either $e \leqslant (1-e)$ so $e = e^2 \leqslant 0$ or $(1-e) \leqslant e$ so $1 - e \leqslant 0$.
2. $ax \leqslant ay \;\Rightarrow\; x \leqslant y$ if $a > 0$, else $a \leqslant 0 \;\Rightarrow\; x \geqslant y$.

   $ax = 0$ $(a \neq 0) \;\Rightarrow\; |x| < 1$ (else $|x| \geqslant 1 \;\Rightarrow\; |a| \leqslant |a||x| = |ax| = 0$).
3. Recall that linear orders have a natural T_5 topology; which is connected iff order-complete and without cuts or gaps.
4. Reduced linearly ordered rings are domains.

3.2.1 Linearly Ordered Fields

Examples:

- $\mathbb{Q}$, $\mathbb{R}$
- $\mathbb{Q}(\sqrt{2})$ with (i) $\sqrt{2} > 0$, (ii) $\sqrt{2} < 0$.
- *Hyperreal numbers*: $\mathbb{R}^{\mathbb{N}}$ with $(a_n) \leqslant (b_n) \;\Leftrightarrow\; \{\, n \in \mathbb{N} : a_n \leqslant b_n \,\} \subseteq \mathcal{N}$, where $\mathcal{N}$ is a maximal non-principal filter of $\mathbb{N}$; sequences need to be identified to give an order. Then $\epsilon := (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ is an infinitesimal with inverse $\omega := (1, 2, 3, \ldots)$. (This field is independent of $\mathcal{N}$ if the continuum hypothesis is assumed.)

1. The prime subfield is $\mathbb{Q}$.
2. $x \mapsto ax$ for $a > 0$ are precisely the $(+, \leqslant)$-automorphisms. The only $(+, *, \leqslant)$-automorphism is trivial.
3. If $x \leqslant y + a$ for all $a > 0$, then $x \leqslant y$ (else $x - y \leqslant a := (x - y)/2$).

4. A field can be linearly ordered $\Leftrightarrow$ it can be lattice-ordered $\Leftrightarrow$ it is formally real.

   Proof: A formally real field can have its positives P extended maximally to Q, by Hausdorff's maximality principle. For $x \notin Q$, $Q - Qx \supseteq Q$ is also a positive set, so $Q - Qx = Q$, i.e., $-x \in Q$.

   More generally a ring can be linearly ordered $\Leftrightarrow$ proper sums of even products of elements cannot be zero (same proof). Note that for a division ring, an even product is a product of squares (since $axay = (ax)^2(x^{-1})^2xy = \cdots$).

5. A linearly ordered field is Archimedean $\Leftrightarrow$ $\mathbb{N}$ is unbounded $\Leftrightarrow$ $\mathbb{Q}$ is dense. ($F \smallsetminus \mathbb{Q}$ is also dense unless empty.)

   Proof: $\forall x,\ x \prec y \Rightarrow \mathbb{N}y$ is unbounded. If $0 \leqslant x < y$ then $(y-x)^{-1} < n$ and $\frac{1}{2n}\mathbb{N}$ is unbounded; pick smallest $\frac{m}{2n} > x$. So $x < \frac{m}{2n} \leqslant x + \frac{1}{2n} < x + \frac{y-x}{2} < y$.

6. The extension field $F(a) \cong F[x]/\langle p\rangle$ (p irreducible) can be linearly ordered, if p changes sign. In particular when

   (a) $a^2 > 0$ in F

   (b) p is odd dimensional

   Proof: Let p be a minimal-degree (m) counterexample, i.e., $F[x]/\langle p\rangle$ is not formally real, so $\sum_n p_n^2 = 0 = pq \pmod p$ with $p_n \neq 0$; q has degree at most $2(m-1) - m = m-2$. Since $p(x)q(x) = \sum_n p_n(x)^2 \geqslant 0$ yet $p(x_1)p(x_2) < 0$, then $q(x_1)q(x_2) < 0$; decompose $q = q_1 \cdots q_r$ into irreducibles, then $q_1(x_1)q_1(x_2) < 0$ say, and $\sum_n p_n^2 = pq = 0 \pmod{q_1}$, still not formally real. If $a^2 > 0$ then $x^2 - a^2$ is irreducible in F and changes sign from 0 to a^2+1. If $p(x) = x^n(1 + a_{n-1}/x + \cdots + a_0/x^n)$ is odd, then for x large enough the bracket is positive, hence $p(x)$ changes sign like x^n.

7. (Neumann) Every linearly ordered division ring can be extended to include $\mathbb{R}$.

Proposition 4

Every Archimedean linearly ordered ring is embedded in $\mathbb{R}$, except $R = 0$.

PROOF: $R+$ is embedded in $\mathbb{R}+$ as lattice groups. The map $x \mapsto a \cdot x$ is a group automorphism on $R+$, hence of the type $x \mapsto r_a x$; let $r_{-a} := -r_a$, then $a \mapsto r_a$ is a group morphism $\mathbb{R}+ \to \mathbb{R}+$, so $r_a = sa$, with $s > 0$, so $x \cdot y = r_x y = sxy$, $r_{x\cdot y} = s(x \cdot y) = sxsy = r_x r_y$, hence $x \mapsto r_x$ is an order-ring embedding. (Thus Archimedean linear rings are characterized by their +-group.)

□

Hence, the only order-complete linearly ordered rings are 0, $\mathbb{Z}$ and $\mathbb{R}$; and the Dedekind-completion of any Archimedean linearly ordered field is $\mathbb{R}$. Recall that these are also Cauchy-complete. (Note: The Dedekind completion of the hyperreal numbers is not closed under $+$, etc.)

3.2.2 Surreal Numbers

Every linearly ordered field is embedded in the surreal numbers.

Construction: A *surreal* number is a mapping from an ordinal number to $2 := \{1, -1\}$. The first few examples are sequences:

	-3	-2	$-\frac{3}{2}$	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	3
2^0								$()$							
2^1				(-1)								(1)			
2^2		$\begin{pmatrix}-1\\-1\end{pmatrix}$				$\begin{pmatrix}-1\\1\end{pmatrix}$				$\begin{pmatrix}1\\-1\end{pmatrix}$				$\begin{pmatrix}1\\1\end{pmatrix}$	
2^3	$\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$		$\begin{pmatrix}-1\\-1\\1\end{pmatrix}$		$\begin{pmatrix}-1\\1\\-1\end{pmatrix}$		$\begin{pmatrix}-1\\1\\1\end{pmatrix}$		$\begin{pmatrix}1\\-1\\-1\end{pmatrix}$		$\begin{pmatrix}1\\-1\\1\end{pmatrix}$		$\begin{pmatrix}1\\1\\-1\end{pmatrix}$		$\begin{pmatrix}1\\1\\1\end{pmatrix}$

The surreal numbers in $2^{\mathbb{N}}$ contain the real numbers, as well as $\omega := (1, 1, \ldots)$, $\epsilon := (1, -1, -1, \ldots)$.

If $A < B$ are sets of surreal numbers then $(A|B)$ is the least surreal number such that $A < x < B$; conversely, $x = (A_x|B_x)$ where

$$A_x := \{ x|_\alpha : \alpha < \mathrm{Dom}(x), x(\alpha) = -1 \},$$
$$B_x := \{ x|_\alpha : \alpha < \mathrm{Dom}(x), x(\alpha) = +1 \}$$

e.g. $0 = (|)$, $3/2 = (1|2)$. For $x = (A, B)$, $y = (C, D)$, let

$$x < y \Leftrightarrow \exists c \in C, x \leqslant c \text{ OR } \exists b \in B, b \leqslant y$$
$$x + y := ((A + y) \cup (x + C) \mid (B + y) \cup (x + D)) \text{ where } A + y := \{ a + y : a \in A \}$$
$$xy := (\{ ay + xc - ac \} \cup \{ by + xd - bd \} \mid \{ ay + xd - ad \} \cup \{ by + xc - bc \})$$
$$\text{where } a \in A, b \in B, c \in C, d \in D$$

Then it can be shown these operations give a field: $0 + x = x$, $1x = x$, negatives $-x = (-B, -A)$, reciprocals exist, etc..

References

1. Henriksen, "A survey of f-rings and some of their generalizations"

2. Steinberg, "Lattice-ordered Rings and Modules"