University of Malta
BSc(Hons.) IT Year I

Introduction to Logic and Logic Programming

Lecture 10

First Order Logic

- Variables
- Quantification
- Uses of Quantification
- Functions and Uniqueness

Department of Computer Science and AI
1988/9

Variables and Quantification

Much of the power of FOL comes from its ability to formalise the following kind of argument:

1. All men are mortal
2. Socrates is a man

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3. Therefore, Socrates is mortal

Recall the predicate/argument structure of an atomic sentence in FOL: is n-ary-predicate-symbol(name-1,name-2,...,name-n)

e.g. `Supports(BlockA,BlockB)`

which when interpreted, denotes that the named individuals are in the relation.

The atomic sentences we have met so far only allow us to make assertions of type (1) and (3). To make assertions of type (1) we need to introduce *variables* and *quantifiers*.

Variables

- Variables are a kind of symbol.

- Syntactically they are terms (so can appear in argument lists).

- Main function of a variable is not to name an object, but rather to function as a place-holder to indicate correspondances between "positions" in a formula.

$$(\text{ Student } \underset{\uparrow}{(x)} \rightarrow \text{ Noisy } \underset{\uparrow}{(x)})$$

- Now that we have variables, we can produce well formed formulas like

Man(x), Mortal(y)

- Question: are these formulas sentences?

Formulas versus Sentences

A formula like Man(x) is **not** a sentence because it does not make an identifiable claim. To make such claims we require *quantifiers* to to "bind" the variables. Quantifiers tell us about the number or quantity of things that satisfy some the conditions within the scope of the quantifier. The two most common ones are

- $\forall$: The **universal quantifier**: all possible things

  $\forall$x Man(x): all x's are men

- $\exists$: The **existential quantifier**: at least one thing

  $\exists$x Man(x): there is at least one man

There are others, e.g. ιx Man(x): there is exactly one man.

Syntax of FOL

Backus-Naur Form

Formula	::=	AtomicSent | Complex $\forall$ Var Formula | $\exists$ Var Formula
Complex	::=	(Formula Con Formula) | $\neg$Formula
Con	::=	$\wedge \mid \vee \mid \rightarrow \mid \leftrightarrow$
AtomicSent	::=	Pred (Args)
Args	::=	Term | Term, Args
Term	::=	Const | Var | Function (Args)

N.B. Const, Var, Function and Predicate are sets of symbols.

Free vs. Bound Variables

If P is a wff and v is a variable, then $\forall$v P and $\exists$v P is a wff and v is bound in P

So in

$$\begin{array}{ccccccc} \forall & x & (\text{ Student} & (x) & \rightarrow \quad \text{Noisy} & (x) &) \\ & \uparrow & & \uparrow & & \uparrow & \end{array}$$

x is bound within the scope of the $\forall$.

A variable which is not bound in P is said to be *unbound* or *free* in P.

$$\exists x \text{ Student}(x) \ \& \ \text{Noisy}(y)$$

A sentence is a wff with no unbound variables.

A sentence makes a claim.

$$\exists x \ (\text{ Student}(x) \ \& \ \text{Noisy}(x))$$

Semantics for the Quantifiers: Satisfaction

- The object A *satisfies* the wff cube(X) if A is a cube
- The object A *satisfies* the wff (cube(x) $\wedge$ large(x)) if A is a cube.

In short:

An object O *satisfies* a wff with one unbound variable if the atomic sentence obtained by substituting O for the unbound variable comes out true.

The notion of satisfaction can be generalised to cover a sequence of objects satisfying a wff containing a sequence of variables, e.g.

(felix, fido) (cat(x) $\wedge$ dog(y))

Semantics of Universal Quantification

$\forall x \mathcal{P}$ is true iff for every possible object bound to x satisfies $\mathcal{P}$.

Hence $\forall$ x Student(x)

has the same truth status as the **conjunction** of all the sentences formed by substituting the name of any object in the domain of discourse for X, i.e.

Student(daniel) &
Student(corinne) &
Student(michal) &
$\vdots$

Existential Quantification

$\exists x \mathcal{P}$ is true iff there is at least one object bound to x that satisfies $\mathcal{P}$.

Hence $\exists$ x Student(x)

One can treat the sentence as a *disjunction* of all the sentences obtained by substituting the variable with the name of any object in the domain, i.e.

$$\begin{array}{l} \text{Student(daniel)} \vee \\ \text{Student(corinne)} \vee \\ \text{Student(michal)} \vee \\ \vdots \end{array}$$

Checkpoint

Translate the following into FOL

1. No even number is prime
2. Every prime is either odd or equal to 2
3. Some prime is even
4. Some prime is not even

Which of these sentences are true?

Answers

1. *No even number is prime.* In other words, if something is an even number, it definitely is not prime, i.e.

   $$\forall x\ (\text{Even}(x) \rightarrow -\text{Prime}(x))$$

   Another way of saying this is there is nothing which is both even and prime, i.e.

   $$-\exists x\ (\text{Even}(x) \wedge \text{Prime}(x))$$

   These two formulations are in fact logically equivalent. Neither of them are true.

2. *Every prime is either odd or equal to 2.* This is straightforward. If something is prime, then there are two alternatives. Remember that the equality relation is written infix:

   $\forall x\ (\text{Prime}(x) \rightarrow (\text{Odd}(x) \vee x{=}2))$

   This is true.

3. *Some prime is even.* Straightforward

   $\exists x\ (\text{Prime}(x) \wedge (\text{Even}(x))$

   True

4. *Some prime is not even.* Straightforward

   $\exists x\ (\text{Prime}(x) \wedge -(\text{Even}(x))$

   True

Common Translation Patterns with Quantified Sentences

$\forall$ is often used in sentences like the following:

Every P is a Q

$$\forall x\ (P(x) \rightarrow Q(x))$$

Whilst $\exists$ is normally used as follows

There is a P which also has property Q.

$$\exists x\ (P(x)\ \&\ Q(x))$$

It is often tempting to translate the latter sentence as

$$\exists x\ (P(x) \rightarrow Q(x))$$

but this means something rather different, being true just in case there is an object which is either not a P or else is a Q: in particular, it is true when there is no object satisfying P(x).

Vacuously True Sentences

Suppose we try to evaluate the sentence

$$\forall x \; (\mathrm{Student}(x) \rightarrow \mathrm{Noisy}(x))$$

in a world where there are no students. Then from the truth table for implication, all instances of the sentence comes out true – hence the universal statement holds.

Any sentence of the form

$$\forall x \; (P(x) \rightarrow Q(x))$$

is vacuously true in a world where the antecendent does not hold.

Translating Complex Noun Phrases

Most of the time we use $\forall$ to translate sentences with "every" or "all"

Every small dog that is at home is happy.

$$\forall x\ (\text{small}(x)\ \&\ \text{Dog}(x)\ \&\ \text{Home}(x) \rightarrow \text{Happy}(x))$$

and we use $\exists$ to translate sentences involving "a"

A small happy dog is at home.

$$\exists x\ (\text{small}(x)\ \&\ \text{happy}(x)\ \&\ \text{Home}(x))$$

However, sometimes "a" has a universal sense, as in

A dog is a kind of mammal.

$$\forall x\ \exists y(\text{Dog}(x) \rightarrow \text{Kind}(x,y)\ \&\ \text{Mammal}(y))$$

Logical Equivalences involving Negation and Quantifiers

- Intuitively:

  If we know that not everything has some property then we know that there is something that does not have that property.

  If we know that it is not the case that something has a property then all things do not have that property.

- Formally:

  $-(P(a_1)\& \ldots \& P(a_n)) \equiv (-P(a_1) \vee \ldots \vee -P(a_n))$

  $-(P(a_1) \vee \ldots \vee P(a_n)) \equiv (-P(a_1)\& \ldots \& -P(a_n))$

- DeMorgan's Law for Quantifiers:

$$\boxed{\begin{array}{l} -\forall x P(x) \equiv \exists x - P(x) \\ -\exists x P(x) \equiv \forall x - P(x) \end{array}}$$

More than One Quantifier

- $\exists y \exists z(\text{Cube}(y)\ \&\ \text{Tet}(z)\ \&\ \text{Leftof}(y,z))$

  Some cube is to the left of some tetrahedron.

- $\forall y \forall z\ ((\text{Cube}(y)\ \&\ \text{Tet}(z)) \rightarrow \text{Leftof}(y,z)))$

  All cubes are to the left of all tetrahedrons.

- The above sentences are in "prenex form". They need not be:

  $\exists y(\text{Cube}(y)\ \&\ \exists z\ (\text{Tet}(z)\ \&\ \text{Leftof}(y,z)))$

  $\forall y(\text{Cube}(y)\ \&\ \forall z(\text{Tet}(z) \rightarrow \text{Leftof}(y,z)))$

- Another tricky point arises with multiple quantifiers

$$\begin{aligned}\forall x \forall y \quad & (\ (\text{Cube}(x)\ \&\ \text{Cube}(y)) \\ & \rightarrow (\text{Leftof}(x,y) \vee \text{Rightof}(x,y)))\end{aligned}$$

Use of the Equality Predicate

The last sentence is supposed to express

for every pair x,y of cubes either x is to the left of y or x is to the right of y.

This is not true of any world containing more than one cube.

To say this we need to explicitly deny the possibility that x and y are the same cube:

$$
\begin{aligned}
\forall x \forall y \quad & (\ (\mathrm{Cube}(x)\ \&\ \mathrm{Cube}(y))\ \&\ x \neq y \\
& \rightarrow (\mathrm{Leftof}(x,y) \vee \mathrm{Rightof}(x,y)))
\end{aligned}
$$

Mixed Quantifiers

$$\forall x(\mathrm{Cube}(x) \to \exists y(\mathrm{Tet}(y) \,\&\, \mathrm{Leftof}(x,y)))$$

Every cube is to the left of a tetrahedron

$$\forall x \exists y(\mathrm{Cube}(x) \to (\mathrm{Tet}(y) \,\&\, \mathrm{Leftof}(x,y)))$$

The order of the quantifiers is important *when the quantifiers are mixed*. Hence, the above sentence expresses a different claim to

$$\exists y \forall x(\mathrm{Cube}(x) \to (\mathrm{Tet}(y) \,\&\, \mathrm{Leftof}(x,y)))$$

Exercise:

- Draw pictures of worlds which make the above sentences true.
- Translate

  Only large objects have nothing in front of them.

  If a cube has something in front of it, then it is small.

Sentences Involving Functions

Let mother be a unary function symbol and Motherof a binary relation symbol. The following say the same thing:

Motherof(Nancy, Max)
mother(Max)=Nancy

A person's mother is always older than the person.

- ∀x Olderthan(mother(x),x)
- ∀x∃y(Motherof(y,x) & Olderthan(y,x))

  (Everyone has at least one mother that is older than they are)
- ∀x∀y(Motherof(y,x) → Olderthan(y,x))

  (If x is the mother of y then x is older than y)

Uniqueness

With the last two sentences we are missing the sense that

"everyone has at least one and at most one mother" that is somehow present in the original sentence that uses the function symbol.

Everyone has at least one mother

$$\forall x \exists y \ \mathrm{Motherof}(y,x)$$

Everyone has at most one mother

$$\forall x \forall y \forall z \quad (\mathrm{Motherof}(y,x) \ \& \ \mathrm{Motherof}(z,x) \rightarrow y=z)$$

Combining the two yields

$$\forall x \exists y \quad (\mathrm{Motherof}(y,x) \ \& \ \forall z(\mathrm{Motherof}(z,x) \rightarrow y=z))$$