University of Malta
BSc(Hons.) IT Year I

Introduction to Logic and Logic Programming

Lecture 11

Quantification

- $\forall$ Introduction & Elimination
- $\exists$ Introduction and Elimination

Department of Computer Science and AI

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Inference Rules for $\forall$ and $\exists$

$\forall$ Elimination

$$
\begin{array}{l|l}
 & \forall x S(x) \\
 & \vdots \\
\triangleright & S(c)
\end{array}
$$

$\exists$ Introduction

$$
\begin{array}{l|l}
 & S(c) \\
 & \vdots \\
\triangleright & \exists x S(x)
\end{array}
$$

Where:

- x is any variable
- c is any individual constant
- S(c) is the result of substituting c for unbound occurrences of x in S(x).

Universal Introduction

$$
\begin{array}{l|l}
 & \begin{array}{|l}
\boxed{c} \\
\hline
\vdots \\
P(c)
\end{array} \\
\triangleright & \forall x \, P(x)
\end{array}
$$

- $\boxed{c}$ stands for the introduction of an arbitrary constant symbol c which may not occur outside the subproof where it is introduced. Informally, it express *let c denote an arbitrary object*.
- When we give the justification for a universal introduction, we cite the introduction of the arbitrary constant symbol together with the last line of the proof.

Example of Universal Introduction

∀x(R(x) → S(x))

∀x R(x)

⊢

∀x S(x)

1. $\forall x(R(x) \rightarrow S(x)$
2. $\forall x R(x)$

$\boxed{\text{d}}$

---

3. $R(d) \rightarrow S(d)$ ∀ elim: 1
4. $R(d)$ ∀ elim: 2
5. $S(d)$ → elim: 3,4

6. ∀x S(x) ∀ intro $\boxed{\text{d}}$ - 5

Existential Elimination

$$
\begin{array}{l|l}
 & \exists x S(x) \\
 & \vdots \\
 & \quad \boxed{\text{c}} \\
 & \quad \text{S(c)} \\
 & \quad \overline{} \\
 & \quad \vdots \\
 & \quad \text{Q} \\
\triangleright & \text{Q}
\end{array}
$$

- Existential Elimination is similar to disjunction elimination, in which we show that Q follows from $P_1 \vee P_2 \vee P_n$ by showing that Q follows individually from each P_i.

- In the case of $\exists$ elimination, we have one case for each actual constant. Insetad of proving each case individually, however, we prove just one case for an arbitrarily chosen consant c.

Example of Existential Elimination

$\forall x(\text{Cube}(x) \rightarrow \text{Large}(x))$

$\forall x(\text{Large}(x) \rightarrow \text{LeftOf}(x, c))$

$\exists x \text{Cube}(x)$

$\vdash$

$\exists x(\text{Large}(x) \wedge \text{LeftOf}(x, c))$

1.	$\forall x(\text{Cube}(x) \rightarrow \text{Large}(x))$	
2.	$\forall x(\text{Large}(x) \rightarrow \text{LeftOf}(x, c))$	
3.	$\exists x \text{Cube}(x)$	
	$\boxed{\text{e}}$	
4.	$\text{Cube}(e)$	
5.	$\text{Cube}(e) \rightarrow \text{Large}(e)$	$\forall$ elim: 1
6.	$\text{Large}(e)$	$\rightarrow$ elim: 5,4
7.	$\text{Large}(e) \rightarrow \text{LeftOf}(e, c)$	$\forall$ elim: 2
8.	$\text{LeftOf}(e, c)$	$\rightarrow$ elim: 6,7
9.	$\text{Large}(e) \wedge \text{LeftOf}(e, c))$	$\wedge$ intro: 6,8
10.	$\exists x(\text{Large}(x) \wedge \text{LeftOf}(x, c))$	$\exists$ intro: 9
11.	$\exists x(\text{Large}(x) \wedge \text{LeftOf}(x, c))$	$\exists$ elim 3, $\boxed{\text{e}}$ -10