University of Malta
BSc(Hons.) IT Year I

Introduction to Logic and Logic Programming

Lecture 4

Logical Equivalence and Logical Implication

- How to Build Truth Tables
- Tautology versus Logical Truth
- Standard Equivalences
- The Substitution Rule
- Manipulating Logical Equivalences

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1988/9

How to Construct a Truth Table for a Complex Sentence

1. Make the first row of the table consisting of all the variables followed by a fully parenthesised version of the formula.
2. Write down n columns that encode the possible interpretations underneath the variables. This yields 2^n rows.
3. Systematically fill in the columns under the connectives

   - Start wirh connectives that apply only to atomic sentences.
   - Once this is done we work on sentences whose main connective has already had its value filled in.
   - Carry on until the the main connective of the sentence is filled in.

Truth Table for a Complex Formula

A	B	C	$(A \wedge B)$	$\vee$	-C
T	T	T	T	T	F
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

- Standard ordering for rows is worth adopting.

- Shortcuts: e.g. if one operand of $\wedge$ is false, there is no need to evaluate the other one.

- Some rows may be spurious: for instance, if A is the sentence 2=2, it can never be false. In such a case the bottom half of the table would be spurious.

Definitions

Let S be a sentence of FOL built up using truth functional connectives

- **S is a tautology** if and only if every row assigns T to S.
- **S is satisfiable** if and only if there is at least one non-spurious row of the truth table that assigns T to S.
- **S is logically true** if and only if every *non-spurious* row assigns T to S.

Exercises

Determine which of the following are (a) tautologies (b) logically true.

1. $(A \wedge B) \vee (-A \vee -B)$
2. $(A \wedge B) \vee (A \wedge -B)$

Repeat the exercise assuming that

3. A is logically true.
4. A is logically false.

Standard Equivalences

For atomic sentences P,Q,R, logically true sentence T, and logically false sentence F:

1	P v P	$\equiv$	P
2	P & P	$\equiv$	P
3	(P v Q) v R	$\equiv$	P v (Q v R)
4	(P & Q) & R	$\equiv$	P & (Q & R)
5	P v Q	$\equiv$	Q v P
6	P & Q	$\equiv$	Q & P
7	P v (Q & R)	$\equiv$	(P v Q) & (P v R)
8	P & (Q v R)	$\equiv$	(P & Q) v (P & R)
9	P v T	$\equiv$	T
10	P & F	$\equiv$	F
11	P v F	$\equiv$	P
12	P & T	$\equiv$	P
13	P v -P	$\equiv$	T
14	P & -P	$\equiv$	F
15	-(P & Q)	$\equiv$	-P v -Q
16	-(P v Q)	$\equiv$	-P & -Q
17	P $\rightarrow$ Q	$\equiv$	-P v Q
18	P $\leftrightarrow$ Q	$\equiv$	(P $\rightarrow$ Q) & (Q $\rightarrow$ P)
19	P $\rightarrow$ Q	$\equiv$	-Q $\rightarrow$ -P
20	--P	$\equiv$	P

Using Standard Equivalences
The Problem

Standard equivalances apply to individual sentences, so that if we have

$P \vee P \equiv P$

it doesn't follow that

$Q \vee Q \equiv Q$

To make it follow we need to invoke

The Substitution Rule

Let $\mathcal{S}$ be a sentence containing the atomic sentence P.

If $\mathcal{S}$ is a tautology, the formula obtained by substituting another formula for P consistently throughout $\mathcal{S}$ is also a tautology.

The Substitution Rule In Operation

Example

Let $\mathcal{S}$ be the tautology P v -P, and let the substitution for P be A v -B

Then by the substitution rule

$$(A \vee -B) \vee -(A \vee -B)$$

is also a tautology

N.B. For the substitution to work, it has to operate upon a tautology.

How can we hook up a tautology to each of our equivalences?

Logical Connectives
Biconditional

- **Syntax**

  If P and Q are sentences, then $(P \leftrightarrow Q)$ is their material biconditional.

- **Semantics**

P	Q	$(P \leftrightarrow Q)$
t	t	t
t	f	f
f	t	f
f	f	t

N.B. $\leftrightarrow$ is NOT a metasymbol but actually a part of FOL. We can therefore write out a truth table for complex sentences involving $\leftrightarrow$.

The Relationship between $\equiv$ and $\leftrightarrow$

Suppose, for two sentences $\mathcal{F}$ and $\mathcal{G}$, we have

$$\mathcal{F} \equiv \mathcal{G}$$

Then clearly, they take the same truth value for all possible interpretatations. For this reason, the *sentence*

$$\mathcal{F} \leftrightarrow \mathcal{G}$$

will always come out true for all possible interpretations, i.e. will be a tautology.

P	Q	-(P v Q)	-P & -Q	-(P v Q) $\leftrightarrow$ -P & -Q
t	t	t	t	t
t	f	f	f	t
f	t	f	f	t
f	f	f	f	t

$\boxed{\text{If } \mathcal{F} \equiv \mathcal{G} \text{ then } \mathcal{F} \leftrightarrow \mathcal{G} \text{ is a tautology.}}$

The Relationship between $\equiv$ and $\leftrightarrow$

Conversely, suppose we have that $\mathcal{F} \leftrightarrow \mathcal{G}$ is a tautology and that $\mathcal{F}, \mathcal{G}$ contain atomic sentences P,Q,R etc.. Then for all possible assignments, the sentence comes out true.

P,Q,R,etc	$\mathcal{F} \leftrightarrow \mathcal{G}$
all possible	t
assignments	t
to P, Q, R	t
$\vdots$	t

Therefore they have the same truth table. So $\mathcal{F} \equiv \mathcal{G}$.

$\boxed{\text{If } \mathcal{F} \leftrightarrow \mathcal{G} \text{ is a tautology then } \mathcal{F} \equiv \mathcal{G}.}$

$$+$$

$\boxed{\text{If } \mathcal{F} \equiv \mathcal{G} \text{ then } \mathcal{F} \leftrightarrow \mathcal{G} \text{ is a tautology.}}$

$$=$$

$\boxed{\mathcal{F} \equiv \mathcal{G} \text{ if and only if } \mathcal{F} \leftrightarrow \mathcal{G} \text{ is a tautology.}}$

Generalising the Identities

Clearly, for each of the above identities there is a corresponding tautology. Hence, to

$$P \vee P \equiv P$$

there corresponds

$$(P \vee P) \leftrightarrow P$$

. From the substitution rule, it follows that

$$(\mathcal{F} \vee \mathcal{F}) \leftrightarrow \mathcal{F}$$

is a tautology for any formula $\mathcal{F}$; therefore

$$(\mathcal{F} \vee \mathcal{F}) \equiv \mathcal{F}$$

Manipulating Identities

- As a result of the substitution rule, the propositional variables appearing in the standard identities can be substituted (consistently) with arbitrary formulas and not just truth values.

- We can use this property to simplify a formula by rewriting it in a stepwise fashion.

- At each step, we use one of the standard identities (with appropriate substitutions) and thus conserve the truth table expressed by the formula.

Example of Simplification
Show P & (P v Q) $\equiv$ P

$\boxed{P \,\&\, (P \vee Q)}$

11 P $\equiv$ P v F $\downarrow$

$\boxed{(P \vee F) \,\&\, (P \vee Q)}$

7 Pv(Q&R)$\equiv$(PvQ)&(PvR) $\downarrow$

$\boxed{P \vee (F \,\&\, Q)}$

10 P & F $\equiv$ F $\downarrow$

$\boxed{P \vee F}$

11 P $\equiv$ P v F $\downarrow$

$\boxed{P}$

$\boxed{\text{-P \& ((R \& S) v (R \& -S)) \& (P v Q)} \equiv \text{-P \& Q \& R}}$

$\boxed{\text{-P \& ((R \& S) v (R \& -S)) \& (P v Q)}}$

8 P & (Q v R) $\equiv$ (P & Q) v (P & R) $\downarrow$

$\boxed{\text{-P \& (R \& (S v -S)) \& (P v Q)}}$

13 P v -P $\equiv$ T $\downarrow$

$\boxed{\text{-P \& (R \& T) \& (P v Q)}}$

12 P & T $\equiv$ P $\downarrow$

$\boxed{\text{-P \& R \& (P v Q)}}$

6 P & Q $\equiv$ Q & P $\downarrow$

$\boxed{\text{R \& -P \& (P v Q)}}$

$$\boxed{\textbf{-P \& ((R \& S) v (R \& -S)) \& (P v Q)} \equiv \textbf{-P \& Q \& R}}$$

$\boxed{\text{R \& -P \& (P v Q)}}$

8 P & (Q v R) $\equiv$ (P & Q) v (P & R) $\downarrow$

$\boxed{\text{R \& ((-P \& P) v (-P \& Q))}}$

14 P & -P $\equiv$ F $\downarrow$

$\boxed{\text{R \& (F v (-P \& Q))}}$

11 P v F $\equiv$ P $\downarrow$

$\boxed{\text{R \& (-P \& Q)}}$

4 (P & Q) & R $\equiv$ P & (Q & R) $\downarrow$

$\boxed{\text{(R \& -P) \& Q}}$

6 P & Q $\equiv$ Q & P $\downarrow$

$\boxed{\text{-P \& R \& Q}}$