University of Malta
BSc(Hons.) IT Year I

Introduction to Logic and Logic Programming

Lecture 5

Logical Implication

- Logical Consequence
- Logical Implication
- Material Implication
- Testing Argument Validity
- Proof by Refutation
- Tableau Method

Department of Computer Science and AI
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Logical Consequence and Valid Arguments

A major concern of Logic is the concept of *logical consequence*, since this the concept that we can use to analyse valid arguments.

We will say that an argument is *logically valid* if the conclusion is a logical consequence of the premises. Alternatively we can say that the premises *logically imply* the conclusion.

What this means is that whenever circumstances satisfy (i.e. make true) the premises, the same circumstances satisfy the conclusion.

John is a fool or I'm a Dutchman
I'm no Dutchman
Therefore, John is a fool

Logical Implication

A sentence $\mathcal{F}$ *logically implies* a sentence $\mathcal{G}$, means that whenever $\mathcal{F}$ takes the value true, so does $\mathcal{G}$. For example

(P & Q) logically implies P

(P & Q) logically implies P v Q

as can be easily verified using truth tables.

P	Q	(P & Q)	(P v Q)
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	f

Note that the converses do not hold: unlike logical equivalence, logical implication is not symmetrical.

Interpretation, Model

An interpretation of a (set of) sentence(s) S is an assignment of truth values to the atomic sentences in (the set) S.

S is satisfiable if there is at least one interpretation which makes (all members of) S true.

An intepretation that satisfies S is called a model of S

Valid Arguments

We can generalise the definition of logical implication to cover *argument forms*, i.e. statements of the form

$$\boxed{\begin{array}{c} P_1 \\ P_2 \\ \ldots \\ P_{n-1} \\ \hline P_n \end{array}}$$

where $P_1 \ldots P_{n-1}$ are premises and P_n is a conclusion.

In particular, the argument is valid if P_n takes the value true for every model of $P_1, P_2, \ldots P_{n-1}$.

If we do have such a valid argument, we use the metasymbol $\models$ to write either

$$\{P_1, P_2, \ldots P_{n-1}\} \models P_n, \text{ or}$$
$$P_1, P_2, \ldots P_{n-1} \models P_n$$

Testing the Validity of Arguments

An argument consists of premises and a conclusion. The argument is logically valid if the premises logically imply the conclusion.

To test the validity of an argument:

1. Translate premises and conclusion into FOL.

2. Construct truth table for premises and conclusion.

3. The argument is logically valid if and only if every interpretation that makes the premises true also makes the conclusion true.

Logical Connectives
Material Implication

- **Syntax**

  If P and Q are sentences, so is

$$P \rightarrow Q$$

- **Semantics**

P	Q	$P \rightarrow Q$
t	t	t
t	f	f
f	t	t
f	f	t

Material Implication Remarks

- **Odd properties**

  If 2+2=5, my name is Jabberwocky

  when the antecedent is false, the implication is always true

- **Non-Commutative**

  $p \rightarrow q$ vs. $p \rightarrow p$

  rain implies clouds vs. clouds imply rain

- **Non-Assocative**

  $(p \rightarrow q) \rightarrow p$ vs. $p \rightarrow (q \rightarrow p)$

  if (Sunday implies free ride) then I'm going

  if Sunday, then (free ride implies I'm going)

Some Logical Equivalents of English Phrases

English	**Logic**
P implies Q	$P \rightarrow Q$
if P then Q	$P \rightarrow Q$
P if Q	$Q \rightarrow P$
P when Q	$Q \rightarrow P$
P only if Q	$P \rightarrow Q$
P if and only if Q	$P \leftrightarrow Q$
unless P, Q	$-P \rightarrow Q$
P unless Q	$-Q \rightarrow P$
either P or Q	(P v Q) & -(P & Q)
neither P nor Q	-(P v Q)

N.B. Take care! These equivalances are highly approximate

Step 1: Formalizing the Argument

if I water my garden then the flowers will grow

$$WA \rightarrow FL$$

if the flowers will not grow then the weeds will grow

$$-FL \rightarrow WE$$

the weeds will grow

$$WE$$

Therefore I do not water my garden

$$-WA$$

The argument to validate is:

$$WA \rightarrow FL,\ -FL \rightarrow WE,\ WE \models -WA$$

Step 2: Constructing the Truth Table

$$WA \rightarrow FL, \text{-}FL \rightarrow WE, WE \models \text{-}WA$$

WA	FL	WE	(WA→FL)	(-FL→WE)	-WA
t	t	t	t	t	f
t	t	f	t	t	f
t	f	t	f	t	f
t	f	f	f	f	f
f	t	t	t	t	t
f	t	f	t	t	t
f	f	t	t	t	t
f	f	f	t	f	t

The argument is not valid.

Exercise

If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned.

(i) Translate these propositions into Propositional Calculus.

(ii) He further claims that from the above it follows that (a) the unicorn is mythical, and (b) the unicorn is horned. Use truth tables to verify whether either of these conclusions is justified.

Relationship between Logical and Material Implication

P	Q	(P & Q)	(P v Q)	(P & Q) $\rightarrow$ (P v Q)
t	t	t	t	t
t	f	f	t	t
f	t	f	t	t
f	f	f	f	t

$\boxed{If \mathcal{F} \text{ logically implies } \mathcal{G},\ \mathcal{F} \rightarrow \mathcal{G} \text{ is logically true}}$

all interps	$\mathcal{F}$	$\mathcal{G}$	$\mathcal{F} \rightarrow \mathcal{G}$
	t	t	t
$\boxed{\text{spurious}}$	t	f	f
	f	t	t
	f	f	t

$\boxed{\text{If } \mathcal{F} \rightarrow \mathcal{G} \text{ is logically true, } \mathcal{F} \text{ logically implies } \mathcal{G}}$

$\boxed{\mathcal{F} \text{ logically implies } \mathcal{G} \text{ iff } \mathcal{F} \rightarrow \mathcal{G} \text{ is logically true}}$

Proof by Refutation

Assume that

$$\boxed{P_1, P_2, \ldots, P_n \models C}$$

i.e. whenever an interpretation makes the premises true, it also makes the conclusion true.

Clearly, the argument would be *invalid* if we could find an intepretation making the premises true and the conclusion false, or equivalently, a model for

$$\{P_1, P_2, \ldots, P_n, -C\}$$

i.e. interpretation which satisfies

- all the premises and
- the *negation* of the conclusion

A *refutation proof* attempts to show that no such interpretation exists, and hence that the argument is valid.

Example of a Tableau Proof

Suppose we have to establish the validity of:

$$(a \vee b), (\neg b \vee c) \models (a \vee c)$$

First, **we negate the conclusion**, writing all the formulae in a column to indicate that they are logically conjoined.

$$\begin{array}{ll} 1. & a \vee b \\ 2. & \neg b \vee c \\ 3. & \neg(a \vee c) \end{array}$$

Next we go about exploring all possible ways to make the conjunction of 1,2 and 3 true, taking one formula at a time.

Tableau Proof 1

1 $\checkmark$	$a \vee b$		
2	$\sim b \vee c$		
3	$\sim(a \vee c)$		
4	a	b	$1, \vee$

Tableau Proof 2

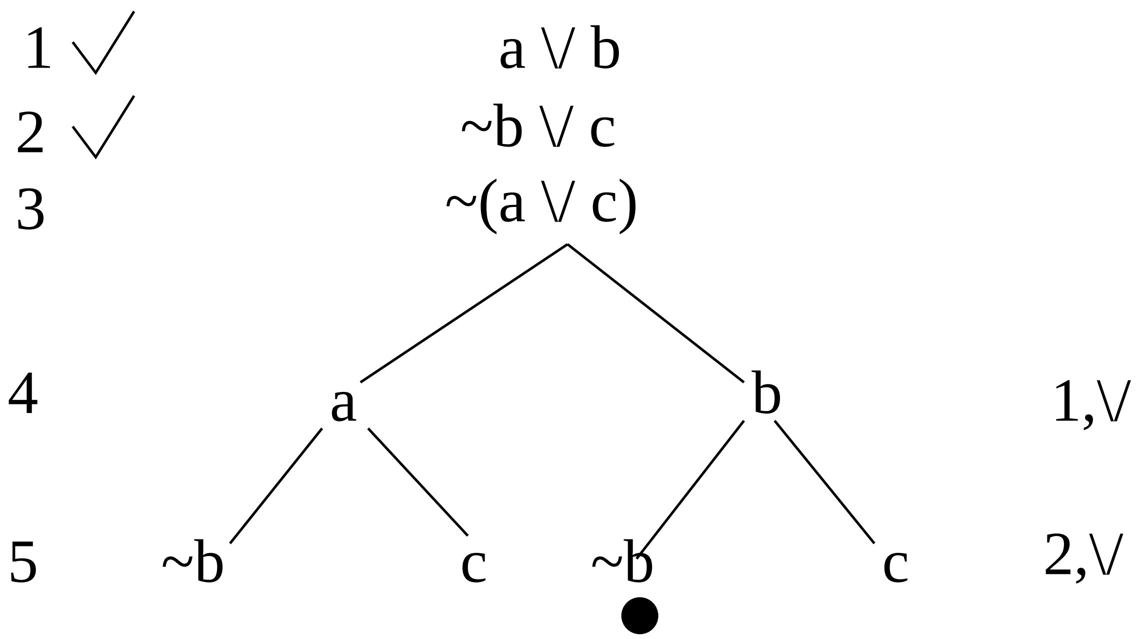
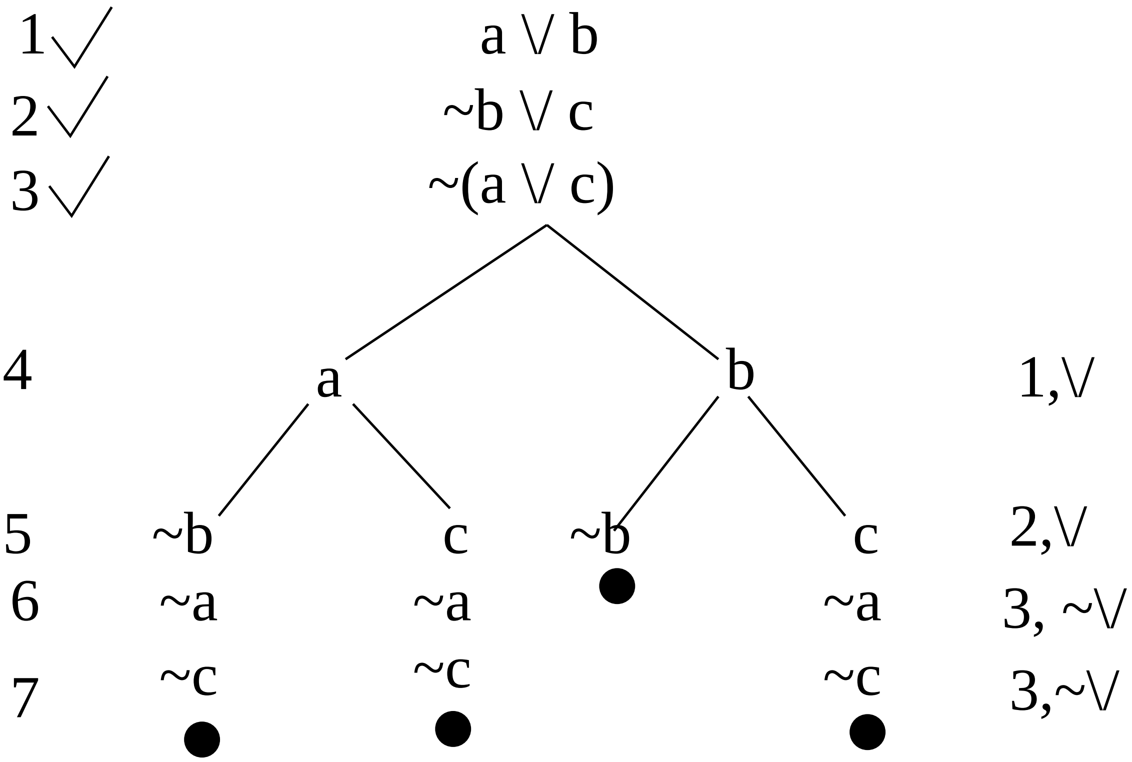


Tableau Proof 3



Stack Rules

$\boxed{\wedge}$ $A \wedge B$

A

B

$\boxed{\sim\vee}$ $\sim(A \vee B)$

$\sim A$

$\sim B$

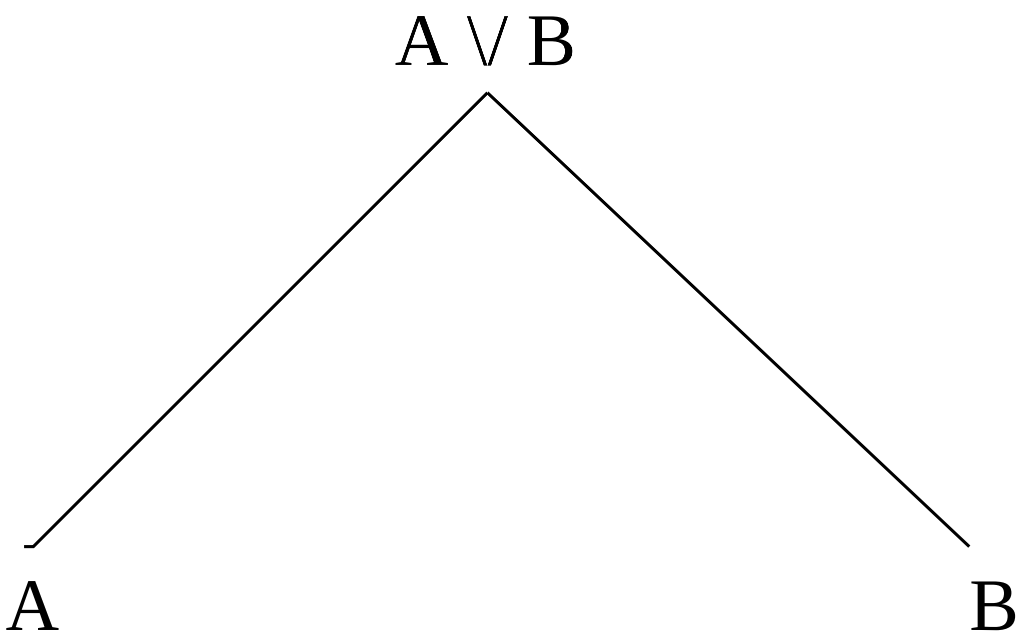
$\boxed{\sim\text{->}}$ $\sim$(A -> B)

A

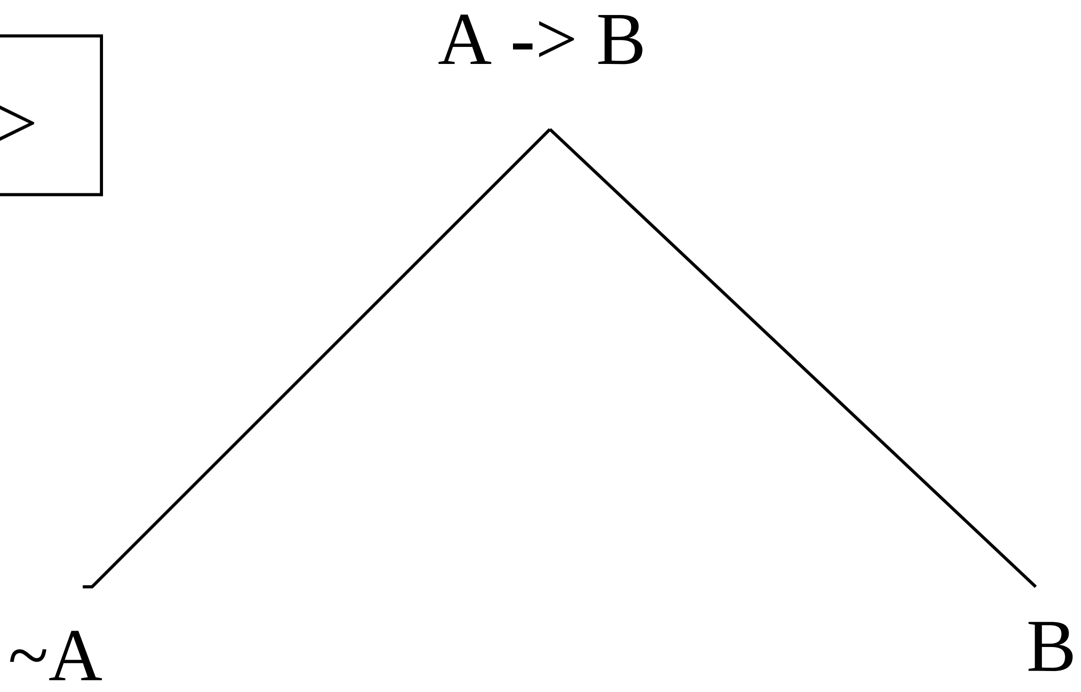
$\sim B$

Branch Rules

$\boxed{\vee}$



$\boxed{\text{->}}$



.Branch Rules

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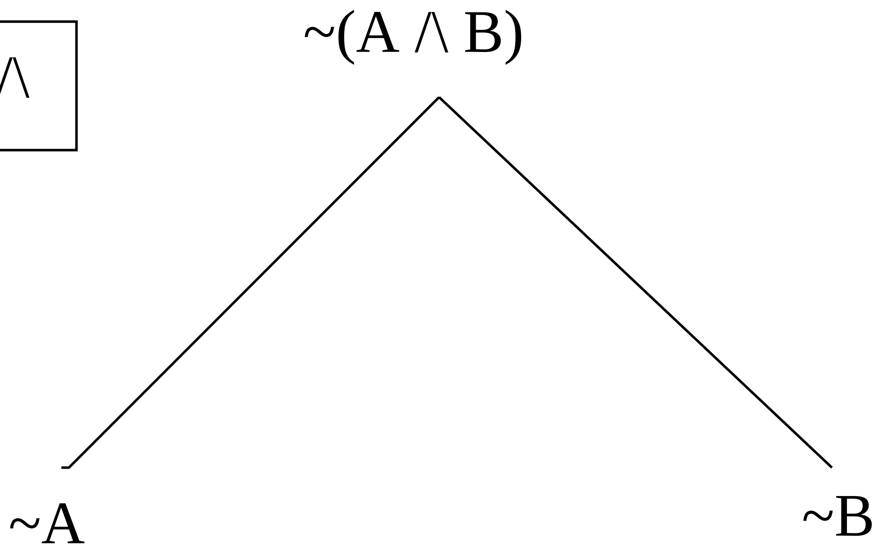

Branch and Stack Rules

<->

A <-> B

A
B

~A
~B

.Branch and Stack Rules

~<->

~(A <-> B)

A
~B

~A
B

Order of Application

The order in which we apply rules does not affect the final result, but it can make a practical difference to the speed with which you get a result.

This means that we are free to choose the order or application to produce the shortest result. Here are some heuristics for shortening the results:

- Work on lines that produce stacks before those which produce branches.

- Work first on branches which will produce closed paths.

- Work on the longest sentence first.