University of Malta
BSc(Hons.) IT Year I

Introduction to Logic and Logic Programming

Lecture 9

Proof Systems

- Negation Rules
- Conditional Proof
- Rules for Conditional

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Natural Deduction
Fitch-Style Systems

A proof in the system $\mathcal{F}$ from premises P,Q,R of a conclusion S takes the following form:

$$
\begin{array}{|l}
\text{P} \\
\text{Q} \\
\text{R} \\
\hline
\vdots \\
\text{S}
\end{array}
$$

Notice:

- The vertical line indicates a single proof comprising several steps
- The horizontal line ("Fitch bar"), indicates a division between *assumptions* and things that follow from them.
- The vertical dots would normally be replaced by intermediate conclusions necessary to derive S.

Example of a Fitch-style Proof

1.	A & B & C	
2.	B	& Elim: 1
3.	C	& Elim: 1
4.	C & B & C	& Intro: 3,2,3

Notice:

- Line numbers have been added so we can refer to proof steps explicitly.
- Each line below the Fitch bar has a justification comprising (a) the name of an inference rule and (b) the lines to which it applies.
- The rule used in lines 2 and 3 is "& Elimination" operating on line 1.
- The rule used in line 4 is "& Introduction". Notice the line numbers it is applied to are repeated to yield the silly conclusion.

Inference Rules: Conjunction Elimination Conjunction Introduction

Conjunction Elimination

$$
\begin{array}{r|l}
 & P_1 \& \ldots \& P_i \& \ldots \& P_n \\
 & \vdots \\
\triangleright & P_i
\end{array}
$$

Conjunction Introduction

$$
\begin{array}{r|l}
 & P_1 \\
 & \Downarrow \\
 & P_n \\
 & \vdots \\
\triangleright & P_1 \& \ldots \& P_n
\end{array}
$$

- The $\Downarrow$ indicates that $P_1 \ldots P_n$ must appear before the conjunction can be asserted and in any order.
- In the justification, the line numbers of the conjuncts should be cited in the order they appear in the conjunction.
- The symbol $\triangleright$ indicates the step which is licenced by the rule.

Disjunction Rules

Disjunction Introduction

$$
\begin{array}{r|l}
 & P_i \\
 & \vdots \\
\triangleright & P_1 \vee \ldots \vee P_i \vee \ldots \vee P_n
\end{array}
$$

Disjunction Elimination

$$
\begin{array}{r|l}
 & P_1 \vee \ldots \vee P_n \\
 & \vdots \\
 & \begin{array}{|l} P_1 \\ \hline \vdots \\ \mathsf{S} \end{array} \\
 & \Downarrow \\
 & \begin{array}{|l} P_n \\ \hline \vdots \\ \mathsf{S} \end{array} \\
 & \vdots \\
\triangleright & \mathsf{S}
\end{array}
$$

Disjunction Elimination introduces the notion of a *subproof* – a proof within a proof which may begin with assumptions and a Fitch bar.

Proof Using Disjunction Elimination

```
| 1. (A & B) v (C & D)
|—
| | 2. A & B
| |—
| | 3. B         & Elim: 2
| | 4. B v D     v Intro: 3
|
| | 5. C & D
| |—
| | 6. D         & Elim: 5
| | 7. B v D     v Intro: 6
|
| 8. B v D                        v Elim 1,2-4,5-7
```

Checkpoint

Give Fitch-style proofs of the following

1. P v Q from the premise P & Q
2. C v B from the premise (A & B) v C
3. (A & B) v (A & C) from the premises A and B v C.

Rules for Negation

Negation Elimination

$$\begin{array}{l|l} & --P \\ & \vdots \\ \triangleright & P \end{array}$$

Negation Introduction

$$\begin{array}{l|l} & P \\ & \begin{array}{|l} P \\ \hline \vdots \\ Q \,\&\, -Q \end{array} \\ \triangleright & -P \end{array}$$

Example prove --A from A

```
| 1. A
|_
| | 2. -A
| |_
| | 3. A & -A         & Intro 1,2
| 4. --A              - Intro 2-3
```

Anything Follows from A Contradiction

Shape of the Proof

```
|  P
| -P
|---
| ⋮
| ⋮
| ⋮
| Q
```

The Proof

```
| 1. P
| 2. -P
|---
| | 3. -Q
| |---
| | 4. P & -P        & Intro 1,2
|
| 5. --Q             - Intro 2-3
| 6. Q               - Elim 5
```

A Bad Proof: Scope Rules

1.	(B & A) v (A & C)	
2.	B & A	
3.	B	& Elim 2
4.	A	& Elim 2
5.	A & C	
6.	A	& Elim 5
7.	A	v Elim 1, 2-4, 5-6
8.	A & B	& intro 7,3

- A subproof may access lines from an outer proof.
- An outer proof may not access lines from a subproof, since they may depend on local assumptions.

Exercise: -P v Q $\vdash$ -(P & Q)

General Shape of the Proof

-P v -Q

—

⋮

⋮

⋮

-(P & Q)

The Method

- Proof by negation
- Assume the negation of the conclusion
- Try to derive a contradiction

The Proof

1. -P v -Q	
2. P & Q	
3. -P	
4. P	& Elim 2
5. P & -P	& Intro 4,3
6. -Q	
7. Q	& Elim 2
8. - (P & -P)	
9. Q & -Q	& Intro 7,6
10. --(P & -P)	- Intro 8-9
11. (P & -P)	- Elim 10
12. (P & -P)	v Elim 1, 3-5, 6-11
13. -(P & Q)	- Intro: 2-12

Summary

	Proof Rules
Valid Steps	 & Elim & Intro v Intro - Elim
Proof Methods Proof by Cases Proof by Contradiction	 v Elim - Intro

Proof Methods: Conditional Proof

Suppose we wish to establish a conditional conclusion, e.g.

$$P \rightarrow Q$$

1. Assume antecedent P
2. Prove conclusion Q
3. Infer conditional $P \rightarrow Q$.

Note that assumption P is "discharged"

Conditional proof in system F corresponds to the deduction theorem in system L.

Conditional Proof - Example

Suppose we wish to establish

$$\text{even}(n^2) \to \text{even}(n)$$

1. Assume antecedent: even(n^2)
2. Prove conclusion even(n) follows
   - assume converse: -even(n)

     $n = 2k + 1$

     $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

     So n^2 is odd

     $\boxed{\text{contradiction}}$
   - Conclude even(n)
3. Infer conditional

Rules for the Conditional

Conditional Elimination (MP)

$$
\begin{array}{l|l}
 & P \rightarrow Q \\
 & \vdots \\
 & \mathrm{P} \\
 & \vdots \\
\triangleright & Q
\end{array}
$$

Conditional Introduction

$$
\begin{array}{l|l}
 & \begin{array}{|l} P \\ \hline \vdots \\ \mathrm{Q} \end{array} \\
 & \\
\triangleright & P \rightarrow Q
\end{array}
$$

Using Conditional Rules

Example $\vdash$A $\rightarrow$ --A

1. A		
2. -A		
3. A & -A	&Intro 1,2	
4. --A	- Intro 1, 2-3	
5. A $\rightarrow$ --A	$\rightarrow$ Intro 2-4	

Example (A v B) $\rightarrow$ C $\vdash$ A $\rightarrow$ C

1. (A v B) $\rightarrow$ C	
2. A	
3. A v B	v Intro 2
4. C	$\rightarrow$ Elim 1,3
5. A $\rightarrow$ C	$\rightarrow$ Intro 2-4

Rules for Biconditional

Biconditional Elimination

$$
\begin{array}{l|l}
 & P \leftrightarrow Q \text{ or } Q \leftrightarrow P \\
 & \vdots \\
 & \mathrm{P} \\
 & \vdots \\
\triangleright & Q
\end{array}
$$

Biconditional Introduction

$$
\begin{array}{l|l}
 & \begin{array}{|l} P \\ \hline \vdots \\ Q \end{array} \\
 & \\
 & \begin{array}{|l} Q \\ \hline \vdots \\ P \end{array} \\
 & \\
\triangleright & P \leftrightarrow Q
\end{array}
$$

Summary

	Proof Rules
Valid Steps	
	& Elim
	& Intro
	v Intro
	- Elim
	$\rightarrow$ Elim (MP)
(Subst. of logical equivalents)	$\leftrightarrow$ Elim
Proof Methods	
Proof by Cases	v Elim
Proof by Contradiction	- Intro
Conditional Proof	$\rightarrow$ Intro
If and Only If Proofs	$\leftrightarrow$ Intro